10th All Strands
03/12/2008

Student Name: ______________________________

Class: ______________________________

Date: ______________________________

Instructions: **Read each question carefully and select the correct answer.**

1. Place the following steps in order to solve $4(x + 3) = 20$

1. Distribute the 4 to $(x + 3)$.
2. Divide both sides by 4.
3. Simplify the answer to $x = 2$.
4. Subtract 12 from both sides of the equation.

A. 4, 2, 1, 3
B. 1, 4, 2, 3
C. 4, 1, 2, 3
D. 1, 2, 4, 3

2. What is the second step in solving the following problem?
"The sum of three times a number and six is twenty-seven."

A. Subtract 6 from both sides of the equation.
B. Add 6 to both sides of the equation.
C. Simplify the answer to $x = 11$.
D. Divide both sides by 3.

3. Five teachers went out to dinner and each ordered a different main dish. The teacher's names were Mr. Bart, Mr. Collins, Mr. Falloon, Mr. Hassel, and Mr. Harring. The main dishes were a veggie burger, fish, chicken, a hot dog, and lasagne. The teacher who had a veggie burger sat in between Mr. Bart and Mr. Harring. The teacher who had chicken doesn't play golf but, the teacher who had the fish, the teacher who had lasagne, Mr. Harring, and Mr. Falloon played golf last Saturday. Mr. Hassel didn't have a hot dog. The teacher who had lasagne helped Mr. Bart with his lesson plans. The teacher who ate fish and Mr. Bart teach the same grade. Mr. Collins is allergic to seafood.

Who ordered the chicken?

A. Mr. Bart
B. Mr. Collins
C. Mr. Falloon
D. Mr. Harring

4. Alex, Dan, Tom, Jeff, and Ruben attend a university and live in Palo Alto, California. Before college, they lived in Baton Rouge, Portland, Chicago, Salt Lake City and Sacramento. Tom and Ruben had never been to the West Coast until they came to the university. Dan has never visited Chicago or Baton Rouge; however, his roommate is from Portland so he is hoping to visit there soon. The student from Salt Lake City, Dan, and Ruben are all on the collegiate football team. The student from Baton Rouge, the student from Salt Lake City, and Alex were roommates. The student from Portland and Alex had Calculus together.

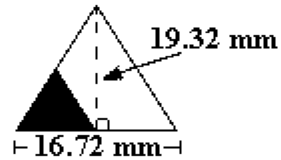
Who was from Portland?

A. Dan
B. Jeff
C. Tom
D. Ruben

5. What is the area of a rectangle that is 7 feet wide and 9 feet long? Remember: $A = l \times w$.

A. 32 ft^2
B. 16 ft^2
C. 61 ft^2
D. 63 ft^2

6. Use the picture to answer the following question.



The area of the shaded region is 1/3 the area of the outer figure. What is the area of the shaded region?

Round your answer to the nearest square millimeter.

A. 108 square millimeters
B. 36 square millimeters
C. 54 square millimeters
D. 72 square millimeters

7. Evaluate the quadratic function at f(-2).

$f(x) = -x^2 - 5x + 12$

A. 18
B. 26
C. 6
D. -2

8. Evaluate the function at $f(9)$.

$$f(x) = -x^2 - 3x + 62$$

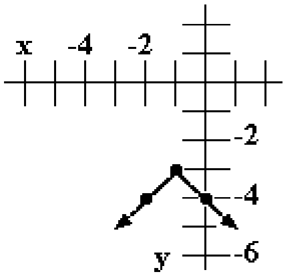
A. - 46
B. 17
C. 8
D. - 170

9. Find the vertex of $y = |2x + 5| + 1$.

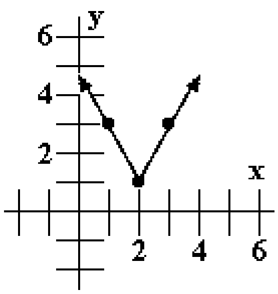
A. (2.5, - 1)
B. (- 5, 1)
C. (- 2.5, 1)
D. (5, - 1)

10. Match the equation y = |2x - 4| + 1 with its graph.

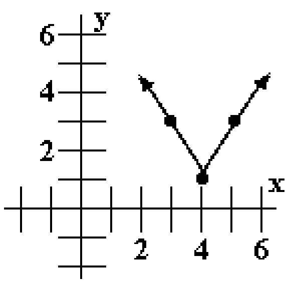
A.



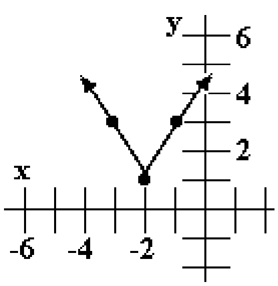
B.



C.



D.



11. Point R and point S have symmetry with respect to the x-axis. What are the coordinates of point S when point R is the point (1, 4)?

A. (1, 4)
B. (1, -4)
C. (-1, -4)
D. (-1, 4)

12. Point X and point Y have symmetry with respect to the y-axis. What are the coordinates of point Y when point X is the point (-3, 5)?

A. (-3, 5)
B. (3, -5)
C. (-3, -5)
D. (3, 5)

13. A triangle has a base equal to 7.1 and height equal to 3.5. A similar triangle has height equal to 26.25. What is the base of the similar triangle?

A. x = 12.94
B. x = 29.75
C. x = 53.25
D. x = 33.35

14. A triangle has a base equal to 5 and height equal to 7. A similar triangle has a base equal to 25. What is the height of the similar triangle?

A. x = 17.9
B. x = 32
C. x = 30
D. x = 35

15. Use Pascal's Triangle to find the binomial coefficient for $\binom{8}{5}$.

A. 120
B. 56
C. 336
D. 3

16. Use Pascal's Triangle to find the binomial coefficient for $\binom{11}{0}$.

A. 0
B. 1
C. 11
D. 39,916,800

17. The area of a rectangle is 56 square feet. The length is (x - 2) feet and the width is (x + 8) feet. What is the width of the rectangle?

A. 20 feet
B. 14 feet
C. 2 feet
D. 4 feet

18. A rectangle has a length that is 8 less than twice its width. The area of the rectangle is 1,050 square feet. What is the length of the rectangle?

A. 42 feet
B. 34 feet
C. 21 feet
D. 25 feet

19. Find the area of a circle with a diameter of 12 m. Express your answer in terms of π.

A. 6π m^2
B. 12π m^2
C. 144π m^2
D. 36π m^2

20. Find the area of the circle whose circumference is 45 inches.

Round your answer to the nearest tenth.

A. 51.3/pi square inches
B. 22.5/pi square inches
C. 506.3/pi square inches
D. 205.1/pi square inches

21. Victoria's parents created a trust fund for her when she was born. Their initial deposit was $315,000.00 in an account that pays 4.1%, compounded continuously, which is modeled by the formula $A = Pe^{rt}$.
How much money will be in the account when Victoria is 25 years old?

A. $328,183.41
B. $340,830.00
C. $642,915.00
D. $877,935.07

22. The amount of money spent by the IRS in millions from 1969 to 1999 can be modeled by the formula below, where 1969 is at t = 3.

$$M(t) = 4,395e^{(0.0067)t}$$

How much money did the IRS spend in 1980?

A. $4,436.42 million
B. $4,424.55 million
C. $4,827.20 million
D. $4,731.15 million

23. Find the characteristic that describes the graph of $y = -4x^2 + x + 5$.

A. The parabola crosses the x-axis at $x = -1.25$ and $x = 1$.
B. The x-coordinate of the vertex is $x = -8$.
C. The axis of symmetry is $x = 0$.
D. The graph opens down.

24. Solve by completing the square.

$$x^2 + 9x + 12 = 0$$

A. $x = -9 \pm \sqrt{69}$

B. $x = -4$ or $x = -3$

C. $x = \dfrac{-9 \pm \sqrt{33}}{2}$

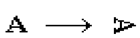
D. No Real Roots

25. Which transformation was performed on the following figure?

L $\longrightarrow$ L

A. Rotation
B. Reflection
C. Translation
D. Dilation

26. Which transformation was performed on the following figure?



A. Rotation
B. Reflection
C. Translation
D. Dilation

27. Which relation is written as a function?

A. $y = \sqrt{x}$

B. $y = x^2 + 1$

C. $y = \sqrt{y}$

D. $y = y^2 + 1$

28. A relation that is NOT a function is expressed on a graph as:

A. a vertical line
B. a diagonal line: top right corner to bottom left corner
C. a diagonal line: top left corner to bottom right corner
D. a horizontal line

29. Find the values for f and r and solve the equation: f(-3) - r(-3)

$$f(x) = -3x + \frac{1}{2}$$
$$r(x) = x^2 - 2$$

A. 9.5, 7
B. 1.5, 1
C. 16.5
D. 2.5

30. Evaluate $f(x)$ and $g(x)$ for the given values to solve the equation, $h = 2f(-3) + 3g(-2)$.

$$f(x) = 5x + 1$$
$$g(x) = x - 5$$

A. $h = 53$
B. $h = -49$
C. $h = 7$
D. $h = -12$

31. A new television show can predict its ratings based on genre, subject matter, star recognition, and time slot. All the factors for the newest show indicate it has a 72% chance of getting good ratings. If a good rating is 11.3 and a bad rating is 4.1, calculate the show's expected rating to the nearest tenth.

A. 11.1
B. 3.0
C. 8.1
D. 9.3

32. The human resources department can estimate the number of positions that will experience turnover in a calendar year based on the economy, the company's success, and the working conditions. All of these factors show that this year has a 78% chance of being a good year. In a good year, 8 positions experience turn over. In a bad year, 27 positions experience turn over. According to this information, how many positions are expected to turn over this year?

A. 12 positions
B. 6 positions
C. 21 positions
D. 22 positions

33. Blood pressures collected from patients taking an experimental medicine are 125, 116, 132, 129, 136, 151, 118, 152, 124, 117, 131, 137, 135, 122, and 158. What is the interquartile range of this set of data?

A. 15
B. 9
C. 6
D. 42

34. What is the standard deviation of the following set of data? Round to the nearest tenth.

5, 8, 12, 17, 11

A. 4.0
B. 10.6
C. 16.2
D. 3.3

10th All Strands
Answer Key
03/12/2008

1.	**B**	Algorithms
2.	**A**	Algorithms
3.	**A**	Logical Reasoning - C
4.	**C**	Logical Reasoning - C
5.	**D**	Accuracy - C
6.	**C**	Accuracy - C
7.	**A**	Functions: Quadratic
8.	**A**	Functions: Quadratic
9.	**C**	Functions: Absolute Value
10.	**B**	Functions: Absolute Value
11.	**B**	Symmetry - D
12.	**D**	Symmetry - D
13.	**C**	Similar Figures - D
14.	**D**	Similar Figures - D
15.	**B**	Binomial Expansion
16.	**B**	Binomial Expansion
17.	**B**	Evaluating Solutions - B
18.	**A**	Evaluating Solutions - B
19.	**D**	Irrational Numbers: Pi
20.	**C**	Irrational Numbers: Pi
21.	**D**	Irrational Numbers: e
22.	**C**	Irrational Numbers: e
23.	**D**	Non-Linear Equations
24.	**C**	Non-Linear Equations
25.	**C**	Spatial Relationships - C
26.	**A**	Spatial Relationships - C
27.	**B**	Functions/Relations - B
28.	**A**	Functions/Relations - B
29.	**D**	Functions: Notation
30.	**B**	Functions: Notation
31.	**D**	Probability
32.	**A**	Probability
33.	**A**	Statistics
34.	**A**	Statistics

Algorithms
An algorithm is a mathematical procedure for solving a problem. The algorithm for solving an equation is performing the order of operations in reverse.

The **order of operations** is just what it sounds like: the order in which one computes operations in an expression. Here is the order of operations:

(1) Parentheses, Brackets, and Braces
(2) Exponents and Roots
(3) Multiply and Divide in order from left to right
(4) Add and Subtract in order from left to right

When solving an equation, use the order of operations to determine the operation you would do first, second, third, etc. then do those operations in reverse order. The goal of solving an equation is to isolate the variable on one side of the equal sign such that the equation ends up being in the form x = n.

Example 1: Solve the problem.
Twice a number, decreased by six equals twenty-four.

Step 1: Write out the equation that represents these words. Replace the name of each number with the correct numeral. Then replace the words "a number" with a variable (x is generally used). Next, replace the words "decreased by" with a subtraction symbol. Finally, replace the word "equals" with an equal sign. You should get the equation:

2x - 6 = 24

Now we need to solve the equation we have written.

(2)	(3)	(4)
$\begin{array}{r} 2x - 6 = 24 \\ +6 \quad +6 \\ \hline 2x = 30 \end{array}$	$\frac{2x}{2} = \frac{30}{2}$	$x = 15$

Step 2: Add 6 to each side of the equation. (The first operation the order of operations tells us to do is multiply 2 and x. Then we would subtract 6 from that answer. Since subtracting 6 is the last step in the order of operations, we need to do the opposite of subtracting 6 first to solve the equation.)
Step 3: Divide each side of the equation by 2. The 2's divide out on the left side of the equation and the x term is isolated on one side of the equation.
Step 4: Simplify the answer to x = 15.

The order of the steps for solving this problem was:
(1) Write the equation, 2x - 6 = 24.
(2) Add 6 to both sides of the equation.
(3) Divide both sides of the equation by 2.
(4) Simplify the answer to x = 15.

Example 2: Solve the problem.
Three times a number increased by the same number is 36.

Step 1: Write out the equation that represents these words. Replace the name of each number with the correct numeral. Then replace the words "a number" and "same number" with a variable ('x' is generally used). Next, replace the words "increased by" with an addition symbol. Finally, replace the word "is" with an equal sign. You should get the equation:

3x +x = 36

Now we need to solve the equation we have written.

(2)	(3)	(4)
$3x + x = 36$ $4x = 36$	$\frac{4x}{4} = \frac{36}{4}$	$x = 9$

Step 2: Add 3x and x together to get 4x. This will begin to isolate the x term on one side of the equation.
Step 3: Divide each side of the equation by 4. The 4's divide out on the left side of the equation and the x term is isolated on one side of the equation.
Step 4: Simplify the answer to x = 9.

The order of the steps for solving this problem was:

(1) Write the equation, 3x + x = 36.
(2) Combine like terms: 3x + x = 4x.
(3) Divide both sides of the equation by 4.
(4) Simplify the answer to x = 9.

Example 3: Solve the following equation.

$$5(x + 4) = 40$$

(1)	(2)	(3)	(4)
$5(x + 4) = 40$ $5 \cdot x + 5 \cdot 4 = 40$ $5x + 20 = 40$	$5x + 20 = 40$ $\underline{\quad -20 \; -20}$ $5x = 20$	$\frac{5x}{5} = \frac{20}{5}$	$x = 4$

Step 1: Distribute the 5. This involves multiplying each term inside the parentheses by 5. Five times x is 5x and 5 times 4 equals 20.
Step 2: Subtract 20 from each side of the equation. This will begin to isolate the x term on one side of the equation.
Step 3: Divide each side of the equation by 5. The 5's divide out on the left side of the equation and the x term is isolated on one side of the equation.
Step 4: Simplify the answer to x = 4.

The order of the steps for solving this problem was:

(1) Distribute the 5 to (x + 4).
(2) Subtract 20 from both sides of the equation.
(3) Divide both sides of the equation by 5.
(4) Simplify the answer to x = 4.

Logical Reasoning - C

Solving logical reasoning problems requires working through the problems slowly and methodically. It may be useful to have the student develop tables and charts to help him or her visualize the problems.

Example 1: Bob, Kenn, Lou, and Marcie drew different pictures: a boat, a kite, a lamb, and a mouse. None of their names begin with the same first letter as their drawings. Marcie and Lou drew animals. What did Kenn draw?

Step 1: Fill the table with the first set of information given: no person's name begins with the same first letter as that person's drawing.

	Boat	Kite	Lamb	M
Bob	No			
Kenn		No		
Lou			No	
Marcie				N

Step 2: Fill the table with the next set of information given: Lou and Marcie drew pictures of animals. If Lou didn't draw a lamb, then Lou must have drawn the mouse. This same logic applies for Marcie, who drew the lamb.

	Boat	Kite	Lamb	Mouse
Bob	No			
Kenn		No		
Lou			No	Yes
Marcie			Yes	No

Step 3: Fill the table with the "logical reasoning." If all four children drew different pictures, then Kenn and Bob did not draw animals.

	Boat	Kite	Lamb	Mouse
Bob	No		No	No
Kenn		No	No	No
Lou			No	Yes
Marcie			Yes	No

Step 4: Complete the table for Kenn. According to the table, Kenn did not draw the kite, the lamb or the mouse.

	Boat	Kite	Lamb	Mouse
Bob	No		No	No
Kenn	Yes	No	No	No
Lou			No	Yes
Marcie			Yes	No

Answer: Kenn drew the boat.

Accuracy - C

Area is the measure, in square units, of the interior region of a two-dimensional figure.

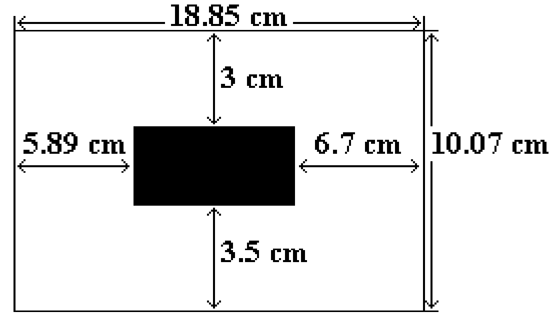
The formula for area of a rectangle is:

Area (A) = length x width.

The formula for the area of a triangle is:

Area (A) = (1/2) x base x height

Example 1: What is the area of the shaded region? Round your answer to the nearest square centimeter.



(1) 18.85 - 6.7 - 5.89 = 6.26 cm

(2) 10.07 - 3 - 3.5 = 3.57 cm

(3) Area = Length x width

(4) Area = 6.26 x 3.57 = 22.3482

(5) 22.3482 ~ 22

Step 1: Determine the length of the shaded region by subtracting 5.89 and 6.7 from 18.85. The length of the shaded region is 6.26 cm.

Step 2: Determine the width of the shaded region by subtracting 3 and 3.5 from 10.07. The width of the shaded region is 3.57 cm.

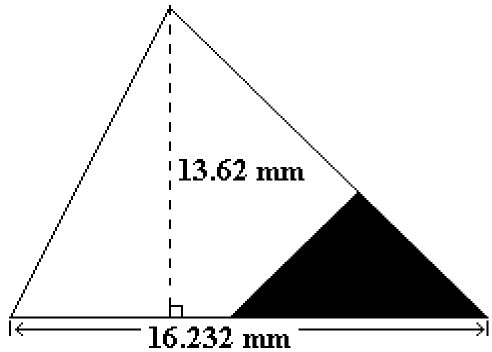
Step 3: Select the appropriate area formula. We are finding the area of a rectangle, so we need the formula for the area of a rectangle.

Step 4: Substitute the values for the length and width of the shaded region into the formula for the area of a rectangle and multiply.

Step 5: The directions required rounding the answer to the nearest square centimeter, so 22.3482 rounds to 22 square centimeters. The ~ symbol means approximately.

Answer: 22 square centimeters

Example 2: The area of the shaded region is 25% of the area of the outer figure. What is the area of the shaded region? Round your answer to the nearest square millimeter.



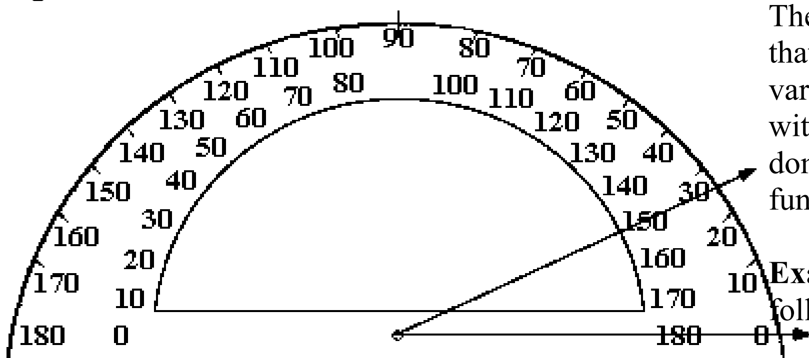
(1). (1/2) x 16.232 x 13.62 = 110.53992
(2). 110.53992 x 25% = 110.53992 x 0.25 = 27.63498
(3). 27.63498 ~ 28

Step 1: Determine the area of the outer triangle using the formula for the area of a triangle. The height of the triangle is 13.62 millimeters and the base of the triangle is 16.232 millimeters. The area of the outer triangle is 110.53992 square millimeters.
Step 2: Since the area of the shaded region is 25% of the area of the outer triangle, multiply the area of the outer triangle by 25%. Before multiplying by 25%, we must convert 25% into a decimal by moving the decimal point two places to the left. 25% equals 0.25.
Step 3: Round the answer to the nearest square millimeter. 27.63498 rounds to 28.

Answer: 28 square millimeters

A protractor is a tool that is used to measure the degree of an angle.

Example 3: What is the measurement of the angle to the closest 5º ?



Solution: Since the angle opens to the right of the protractor, we are going to use the set of numbers on the outside of the protractor. The measure of the angle is between 20º and 30º and just a bit past 25º . Therefore, we would round the measure to 25º .

Functions: Quadratic

A quadratic function is a function that contains polynomial expressions for which the highest power of the unknown variable is two.

Quadratic functions are written in the form:

$$y = ax^2 + bx + c$$

or

$$f(x) = ax^2 + bx + c$$

f(x) is read "f of x."

Here are a few examples of quadratic functions:

(1) $f(x) = x^2 + 3x - 4$
(2) $g(x) = x^2 + 6$
(3) $y = 5x^2 - 2x$

Since the value of f(x) (or y) depends on the value of x, the dependent variable is f(x) and the independent variable is x.

Domain and Range:

The domain of a function is the set of values that can be substituted for the independent variable (x) of a function. When dealing with the coordinate pairs of a function, the domain is the set of all x values of the function.

Example 1: Find the domain of the following function.

$\{(5,2), (3,4), (2,6), (8,14), (1,5), (-2,-$

Since the domain deals with all of the x values of a function, we need to determine all of the x values of this function. The domain of the function is:
$D = \{ -2, 1, 2, 3, 5, 8 \}$.

The domain of a function is generally written in numerical order from smallest to largest.

Example 2: Find the domain of
$g(x) = 3x^2 + 7x - 2$.

Since any real number that is substituted in place of the independent variable (x) will result in another real number, the domain of this function is: D = The set of all real numbers.

The range of a function is the set of values that can result from the substitutions for the independent variable. When dealing with the coordinate pairs of a function, the range is the set of all y values of the function.

Example 3: Find the range of the following function.

$\{(5,2), (3,4), (2,6), (8,14), (1,5), (-2,-3)\}$

Since the range deals with all of the y-values of a function, we need to determine all of the y values of this function. The range of the function is $R = \{-3, 2, 4, 5, 6, 14\}$.
The range of a function is generally written in numerical order from smallest to largest. If a y-value is repeated in an ordered pair, it is only listed once in the range.

Example 4: Find the range of
$g(x) = 3x^2 + 7x - 2$.
Since any real number that is substituted in place of the independent variable (x) will result in another real number (g(x)), the range of this function is: R = The set of all real numbers.

Evaluating Quadratic Functions:

In mathematics, evaluate means to substitute values in for the variables and calculate a result. So, to evaluate a quadratic function, we need to substitute a value in for the independent variable (x) and calculate the result.

Example 5: Evaluate the quadratic function at f(3).

$$f(x) = 2x^2 + 3x - 8$$

(1) $f(3) = 2(3)^2 + 3 \bullet 3 - 8$
(2) $f(3) = 2(9) + 9 - 8$
(3) $f(3) = 18 + 9 - 8$
(4) $f(3) = 19$

Step 1: The directions told us to evaluate the quadratic function at f(3), so we substitute 3 in place of all of the x's in the quadratic function.
Step 2: Following the order of operations, we perform operations on exponents first, then we multiply.
Step 3: Once again, we follow the order of operations which states that multiplying comes before adding and subtracting.
Step 4: The final step in the order of operations states that adding and subtracting are completed by working from the left to the right. First we will add 18 and 9. Then we subtract 8 from that sum. We now know that f(3) = 19.

It is possible to evaluate a function and still have a variable in the answer.

Example 6:
If $f(x) = (x+3)^2 + 5$, find $f(x+4)$.

$$(1)\ f(x+4) = (x+4+3)^2 + 5$$
$$(2)\ f(x+4) = (x+7)^2 + 5$$
$$(3)\ f(x+4) = (x+7)(x+7) + 5$$
$$(4)\ f(x+4) = x^2 + 7x + 7x + 49 + 5$$
$$(5)\ f(x+4) = x^2 + 14x + 54$$

Step 1: Substitute (x + 4) in place of all of the x's in the quadratic function.
Step 2: Applying the order of operations, we must do all operations inside the parentheses. So we add 4 and 3.
Step 3: To take (x + 7) to the second power, we must multiply (x + 7) by itself.
Step 4: In order to multiply (x + 7) by itself, we use the FOIL method. Multiply the "first" terms together
(x times x). Then multiply the "outer" terms (x times 7). Next, multiply the "inner" terms (7 times x). Finally, multiply the "last" terms (7 times 7).
Step 5: Since 7x and 7x are like terms, we can add them together to get 14x. Also, 49 and 5 are like terms, so we can add them together to get 54.

Compositions of Quadratic Functions:

The composition of functions is the operation of first applying one function, then applying the other. Compositions of functions are denoted in two ways. One of those ways is $(g \circ f)(x)$, which can be read, "the composite of g and f of x." The other way is g(f(x)), which can be read, "g of f of x."

Example 7:
Find $g(f(x))$ when $f(x) = x^2 + 2x + 3$ and $g(x) = 6x$.

$$(1)\ g(x^2 + 2x + 3) = 6(x^2 + 2x + 3)$$
$$(2)\ g(f(x)) = 6x^2 + 12x + 18$$

Step 1: Since we are trying to find "g of f of x," we can substitute $(x^2 + 2x + 3)$ in place of f(x) in g(f(x)) to get $g(x^2 + 2x + 3)$. Then, we can substitute that term in place of x in the g(x) = 6x function.
Step 2: Use the distributive property to multiply each term in the parentheses by 6.

Example 8: Find f(g(x)) if f(x) = 2x + 6 and g(x) = 7x - 2.

(1) f(7x - 2) = 2(7x - 2) + 6
(2) f(7x - 2) = 14x - 4 + 6
(3) f(g(x)) = 14x + 2

Step 1: Substitute (7x - 2) in place of g(x) to get f(g(x)) = f(7x - 2). Then substitute (7x - 2) in place of all x's in the f(x) = 2x + 6 function.
Step 2: Use the distributive property to multiply 7x and -2 by 2.
Step 3: Since -4 and 6 are like terms, they can be added together to make 2. So, f(g(x)) = 14x + 2.

Compositions of functions can be evaluated if we are given the value of the variable.

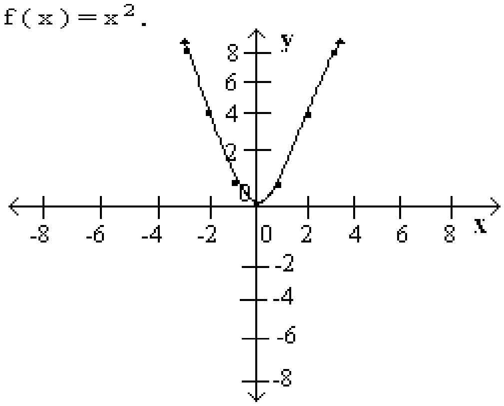
Example 9:
For $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, find $f(g(4$

$$(1)\ g(4) = 4^2 + 1$$
$$(2)\ g(4) = 17$$
$$(3)\ f(g(17)) = 2(17) - 3$$
$$(4)\ f(g(17)) = 31$$

Step 1: To determine the value of f(g(4)), we must first evaluate g(4). We substitute 4 in place of x in $g(x) = x^2 + 1$. Step 2: Following the order of operations we square 4 (4 x 4 =16) and add one to the answer. g(4) = 17.
Step 3: Now we substitute 17 in place of g(4) to get f(g(4)) = f(17). Then substitute 17 in place of x in
f(x) = 2x - 3.
Step 4: Following the order of operations we multiply 2 and 17 (34), then subtract 3.

Comparing Graphs of Quadratic Functions:

A parabola is the shape of the graph of $f(x) = x^2$.



The standard form for the equation of a parabola is $f(x) = a(x - h)^2 + k$. (Please remember that f(x) and y can be substituted for each other in quadratic functions.) When changes are made to an equation such as adding a constant, changes in the position of the graph of the equation will take place. If the value of a < 0, the graph of the function will reflect itself across the x-axis. The value of h determines whether the graph will shift right or left and the value of k determines whether the graph will shift upward or downward. See the chart below.

Variable	Result to Graph
$a < 0$	graph reflects itself across the x-axis
$h > 0$	graph shifts to the right $\|h\|$ units
$h < 0$	graph shifts to the left $\|h\|$ units
$k > 0$	graph shifts upward $\|k\|$ units
$k < 0$	graph shifts downward $\|k\|$ units

Example 10:

Compare the graph of $g(x) = -(x+4)^2 + 7$ to the graph of $f(x) = x^2$.

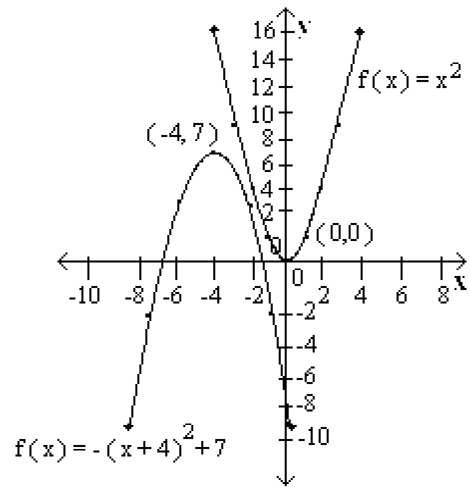
(1)

$g(x) = -(x+4)^2 + 7$ can be rewritten in standard form as $g(x) = -(x-(-4))^2 + 7$.

(2) h = -4, k = 7, and a = -1

(3) By looking at the chart, we can determine that the graph of should shift to the left 4 units (h = -4), upward 7 units (k = 7), and should reflect itself across the x-axis (a = -1).

(4) Compare the actual graphs.



Functions: Absolute Value

The absolute value of a number is the positive distance between the number and 0 on a number line. Absolute value is denoted by one straight vertical line on each side of a number or expression. Here are some examples of absolute value notation.

(1) |-6| = 6
(2) |2| = 2
(3) |0| = 0
(4) |x - 3|
(5) |2x + 6|

The absolute value of the first three examples can be determined because they are all integers. Remember the absolute value of a number is the positive distance between the number and zero on a

number line.

An absolute value function is a function which contains at least one absolute value expression. An absolute value function can be evaluated if the value of the variable is known by substituting the value in for the variable and calculating the result.

Evaluating Absolute Value Functions:

Example 1: Evaluate y = |x - 7| + 3, for x = 9.

(1) y = |9 - 7| + 3
(2) y = |2| + 3
(3) y = 2 + 3
(4) y = 5

Step 1: Substitute 9 in place of x in the absolute value function.
Step 2: Before the absolute value can be figured, 7 must be subtracted from 9. This is a rule: All operations inside an absolute value symbol must be completed before the absolute value can be taken.
Step 3: The distance between 0 and 2 is 2 units, so the absolute value of 2 is 2.
Step 4: The final step is to add 2 and 3 to get 5.

Example 2: Evaluate y = 3|x - 2| + |2x + 4|, for x = -3.

(1) y = 3|(-3) - 2| + |2(-3) + 4|
(2) y = 3|-5| + |-2|
(3) y = 3(5) + 2
(4) y = 15 + 2
(5) y = 17

Step 1: Substitute -3 in place of x in the absolute value function.
Step 2: Compute the values inside both absolute value symbols first.
-3 - 2 = -5 and 2(-3) + 4 = -6 + 4 = -2
Step 3: The distance between 0 and -5 is 5 units, so the absolute value of -5 is 5. The distance between -2 and 0 is 2 units, so the absolute value of -2 is 2.
Step 4: Three times five is 15.
Step 5: Add 15 and 2 to get 17.

Writing an Absolute Value Function as a Compound Function:

Every absolute value function can be written as a compound function. A compound function is a function that is made up of two or more functions. It is very similar to a compound word in language arts. In order to write an absolute value function as a compound function, it is necessary to remember that
$|x| = x \text{ if } x \geq 0 \text{ and } |x| = -x \text{ if } x < 0.$
An example of this rule would be:
When x = 3, |3| = 3; since $3 \geq 0$
When x = -3, |-3| = -(-3) = 3; since -3 < 0

Example 3: Write the absolute value function, f(x) = |2x + 6|, as a compound function.

(1)

$$2x + 6 = 0$$

(2)

$$\begin{array}{r} 2x + 6 = 0 \\ -6 \quad -6 \\ \hline 2x = -6 \end{array}$$

(3)

$$\frac{2x}{2} = \frac{-6}{2}$$
$$x = -3$$

(4)

$$f(x) = \begin{cases} 2x + 6, \text{if } x \geq -3 \\ -2x - 6, \text{if } x < -3 \end{cases}$$

Step 1: In order to write an absolute value function as a compound function, it is necessary to determine the value of x that makes the absolute value function equal to 0. To do this, we set the expression inside the absolute value symbol equal to zero.

Step 2: Solve 2x + 6 = 0. Subtract 6 from each side of the equal sign to isolate the 2x on one side of the equal sign.

Step 3: Divide by 2 on each side of the equal sign to isolate the x on one side of the equal sign. If -3 is substituted into f(x) = |2x + 6|, f(x) = 0.

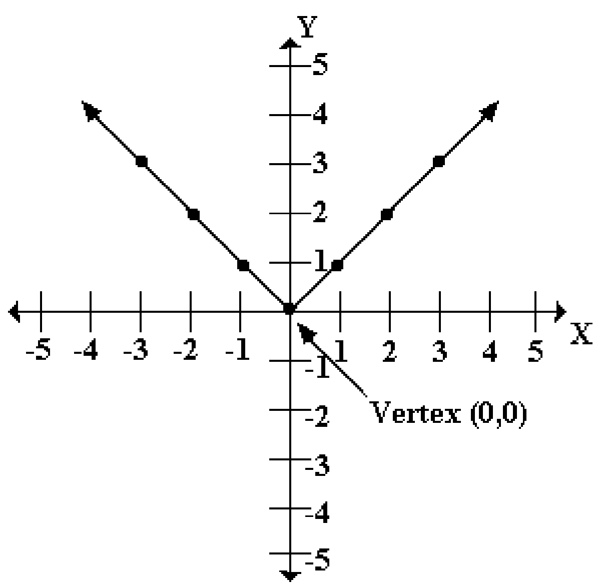
Step 4:

We use $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$ to write the compound function.

x would be 2x + 6, so -x would be -(2x + 6) = -2x - 6. We use the conditions $x \geq -3$ and $x < -3$ because -3 is the value that makes f(x) = |2x + 6| equal zero.

Finding the Vertex of an Absolute Value Function:

A vertex is a point at which two line segments, lines, or rays meet to form an angle. The graph of every absolute value function has a vertex. The function f(x) = |x| is graphed below and the vertex is marked.Remember, the terms f(x) and y can be substituted for each other, so f(x) = |x| can be written as y = |x|.



The standard form for an absolute value function is y = a|bx + c| + d. It is not

necessary to graph an absolute value function to determine the vertex of the function. If the absolute value function is written in standard form, we can use the standard form to determine the vertex. The vertex of an absolute value function is always written as a coordinate point. The x-coordinate of the vertex is where $bx + c = 0$. The y-coordinate of the vertex is the value of d.

Example 4: Find the vertex of $y = |x - 3| + 2$.

(1) $x - 3 = 0$
(2) $x = 3$
(3) $d = 2$
(4) vertex is (3, 2)

Step 1: Set the expression inside the absolute value symbols ($bx + c$) equal to zero and solve to determine the x-coordinate of the vertex.
Step 2: Add 3 to each side of the equal sign to isolate the x and determine that $x = 3$. The x-coordinate of the vertex is 3.
Step 3: Since the 2 in $y = |x - 3| + 2$ is in the same place as the d in the standard form for an absolute value function, $d = 2$. The y-coordinate of the vertex is 2.
Step 4: Put the two coordinates together to make an ordered pair (x, y). The vertex of $y = |x - 3| + 2$ is
(3, 2).

Example 5: Find the vertex of $y = -2|4x + 8| - 6$.

(1) $4x + 8 = 0$
(2) $4x = -8$
(3) $x = -2$
(4) $d = -6$
(5) vertex is (-2, -6)

Step 1: Set the expression inside the absolute value symbols ($bx + c$) equal to zero and solve to determine the x-coordinate of the vertex.
Step 2: Subtract 8 from each side of the equal sign to isolate the 4x on one side of the equal sign.
Step 3: Divide each side of the equation by 4 to get the x by itself on one side of the equal sign. Since
$x = -2$, the x-coordinate of the vertex is -2.
Step 4: Since the -6 in $y = -2|4x + 8| - 6$ is in the same place as the d in the standard form for an absolute value function, $d = -6$. The y-coordinate of the vertex is -6.
Step 5: Put the two coordinates together to make an ordered pair (x, y). The vertex of $y = -2|4x + 8| - 6$ is (-2, -6).

Comparing Graphs of Absolute Value Functions:

The standard form for an absolute value function ($y = a|bx + c| + d$) is needed to compare graphs of absolute value functions with the graph of $y = |x|$. The variable a in the standard form determines whether the graph opens up or opens down. The variables b and c determine whether the graph shifts to the right or the left and the variable d determines whether the graph shifts upward or downward. See the chart below.

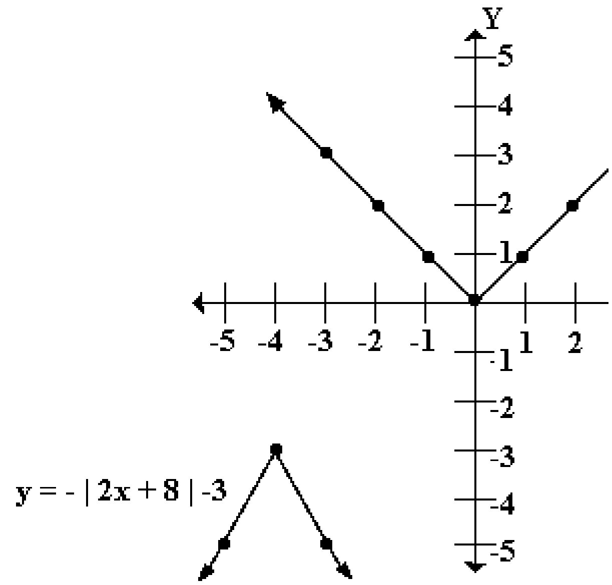
Variable	Result to Graph
$a < 0$	Graph opens downward.
$\frac{c}{b} < 0$	Graph shifts to the right $\left\lvert\frac{c}{b}\right\rvert$ units.
$\frac{c}{b} > 0$	Graph shifts to the left $\left\lvert\frac{c}{b}\right\rvert$ units.
$d < 0$	Graph shifts downward $\lvert d\rvert$ units.
$d > 0$	Graph shifts upward $\lvert d\rvert$ units.

Example 6: Compare the graph of $y = |x|$ with the graph of $y = -|2x + 8| - 3$.

(1) In $y = -|2x + 8| - 3$, $a = -1$, $b = 2$, $c = 8$, and $d = -3$. Refer back to the standard form of an absolute value function to see the placements of a, b, c, and d.

(2) By looking at the chart above, we can see that the graph should <u>open downward</u> ($a = -1$, which is less than 0). To determine whether the graph shifts left or right, we need to determine $c \div b = 8 \div 2 = 4$. Since 4 is greater than zero, the graph will <u>shift to the left 4 units</u>. The last piece of information we have is that $d = -3$. The chart above states that if $d < 0$, the graph will shift $|d|$ units downward, so this graph will <u>shift 3 units downward</u> since $|-3| = 3$.

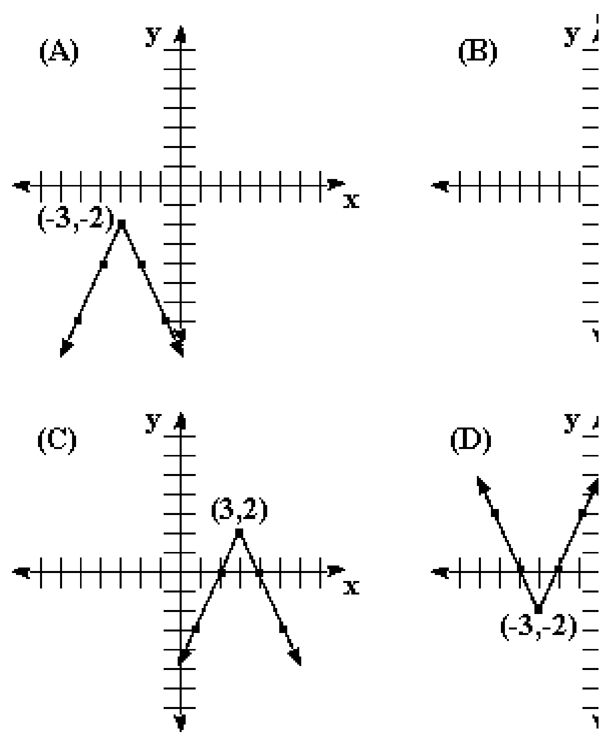
(3) Compare the actual graphs of $y = |x|$ and $y = -|2x + 8| -3$.



Matching an Absolute Value Function (Equation) With its Graph:

It is not necessary to graph an absolute value function to match it with its graph. All that is needed is the standard form for an absolute value function: $y = a|bx + c| + d$.

Example 7: Match the equation $y = -|3x - 9| + 2$ with its graph below.



(1) The first step in matching the equation of

an absolute value function with its graph is to determine the vertex of the graph. The vertex of this graph would be (3, 2) (refer to **Example 4** and **Example 5** for an explanation about determining the vertex of an absolute value function).

(2) Now that the vertex is determined, we can eliminate any graph that does not have (3, 2) as its vertex. Graphs A and D are eliminated.

(3) The next step in matching the equation to its graph is to look at the value of a in the equation.
In $y = -|3x - 9| + 2$, $a = -1$. We learned earlier that if $a < 0$, the graph would reflect itself over the x-axis (in other words, it will open downward).

(4) Since $a = -1$ in the equation we are discussing, we can eliminate any graph that does not open downward. Graph B is not the right graph (D would also be out now if it had not already been disqualified in step 2).

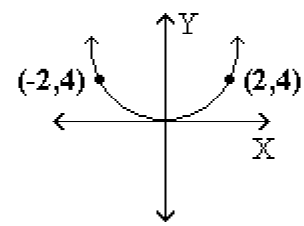
(5) The only graph left is graph C and it is the correct answer. If there were still two (or more) graphs left, it would be necessary to substitute some values in for x to determine the value of y and see which of the graphs contained the calculated points.

Symmetry - D

The graph of an equation can be symmetric about a line or a point. The lines of symmetry that are important to consider when graphing are the x- and y-axes. Often, the origin is the point of symmetry that is considered when graphing a specific equation. Knowledge about the symmetric properties of a graph is helpful to speed up the actual process of graphing an equation.

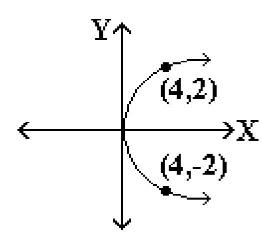
Symmetry with respect to the x- and y-axes:

The graph of an equation is symmetric with respect to the y-axis if both ordered pairs (x, y) and (-x, y) are solutions of the equation. That is, if the point (2, 4) lies on the graph of an equation that is symmetric with respect to the y-axis, then (-2, 4) must also lie on the graph. Pictorially, a graph that is symmetric with respect to the y-axis can be "folded" along the y-axis such that each point that lies to the right of the y-axis will coincide with its corresponding point to the left of the y-axis. An example of a graph that is symmetric with respect to the y-axis is shown below.



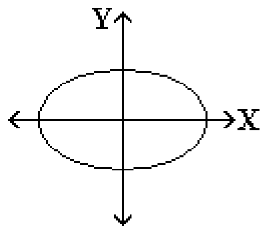
Note that for any point that is chosen that lies to the right of the y-axis, such as (2, 4), its corresponding point, (-2, 4), also lies on the graph.

The graph of an equation is symmetric with respect to the x-axis if both ordered pairs (x, y) and (x, -y) are solutions of the equation. That is, if the point (2, 4) lies on the graph of an equation that is symmetric with respect to the x-axis, then (2, -4) must also lie on the graph. Pictorially, a graph that is symmetric with respect to the x-axis can be "folded" along the x-axis such that each point that lies above the x-axis will coincide with its corresponding point below the x-axis. An example of a graph that is symmetric with respect to the x-axis is shown below.

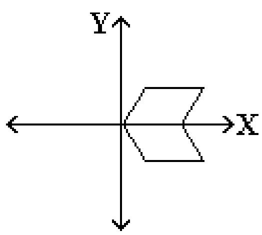


Note that for any point that is chosen that lies above the x-axis, such as (4, 2), its corresponding point, (4, -2), also lies on the graph.

A figure or equation can be symmetric with respect to both axes. The following figure is symmetric about the y-axis because the graph to the right of the y-axis is the mirror image of the graph to the left of the y-axis. The graph of the following figure is also symmetric about the x-axis since the graph above the x-axis is the mirror image of the graph below the x-axis.

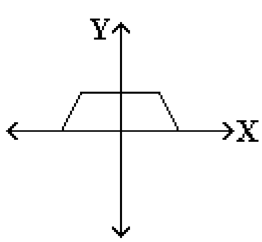


Example 1: Is the following figure symmetric with respect to the x-axis, y-axis, both axes, or neither of the axes?



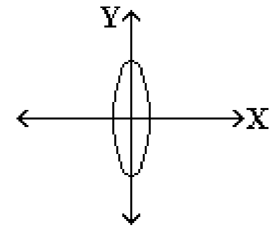
Because each half of the graph is a mirror image with respect to the x-axis, the equation exhibits symmetry with respect to the x-axis.

Example 2: Is the following figure symmetric with respect to the x-axis, y-axis, both axes, or neither of the axes?



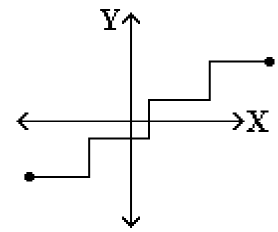
Because each half of the graph is a mirror image with respect to the y-axis, the equation exhibits symmetry with respect to the y-axis.

Example 3: Is the following figure symmetric with respect to the x-axis, y-axis, both axes, or neither of the axes?



Because each half of the graph is a mirror image with respect to both the x- and y-axes, the equation exhibits symmetry with respect to both the x- and y-axes.

Example 4: Is the following figure symmetric with respect to the x-axis, y-axis, both axes, or neither of the axes?



Because neither half of the graph is a mirror image about either of the axes, the figure does not exhibit x-axis or y-axis symmetry.

Example 5: Point A and point B have symmetry with respect to the x-axis. What are the coordinates of point B when point A is the point (3, -5)?

To find the coordinates of point B, simply find the opposite of the given y-coordinate of point A. The opposite of -5 is 5. Do not change the x-coordinates because points that are symmetric with respect to the x-axis have the same x-coordinate. Thus point B is (3, 5).

Example 6: Point A and point B have symmetry with respect to the y-axis. What are the coordinates of point B when point A is the point (3, -5)?

To find the coordinates of point B, simply find the opposite of the given x-coordinate of point A. The opposite of 3 is -3. Do not change the y-coordinates because points that are symmetric with respect to the y-axis have the same y-coordinate. Thus point B is (-3, -5).

Symmetry with respect to a point:

If two points, A and B, are symmetric to a given point, C, then the given point, C, is the midpoint of the line segment that joins points A and B. The midpoint formula given below is used to determine the coordinates of the midpoint of the line segment joining the two points (x_1, y_1) and (x_2, y_2).

$$M(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 7: The points A and B have symmetry with respect to a point C. If point A is (4, 6) and point B is (-8, 10), what are the coordinates of point C?

(1)

$$M(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Step 1: Since points A and B are symmetric with respect to C, C is the midpoint of segment AB. Write down the midpoint formula.

(2)

Let $(x_1, y_1) = (4, 6)$

Let $(x_2, y_2) = (-8, 10)$

$$M(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$M(x_m, y_m) = \left(\frac{4 + -8}{2}, \frac{6 + 10}{2}\right)$$

Step 2: Substitute the given values: $x_1 = 4, y_1 = 6, x_2 = -8, y_2 = 10.$

(3)

$$M(x_m, y_m) = \left(\frac{-4}{2}, \frac{16}{2}\right)$$

$$M(x_m, y_m) = (-2, 8)$$

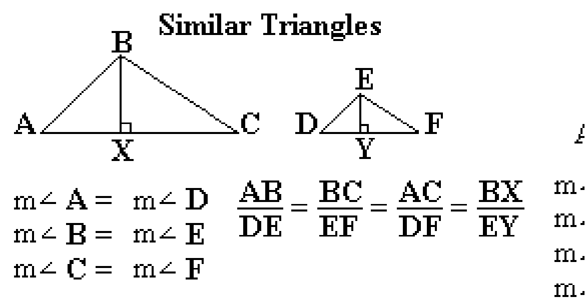
Step 3: Simplify the fractions to get the coordinates of the ordered pair.

Therefore point C is (-2, 8).

Similar Figures - D

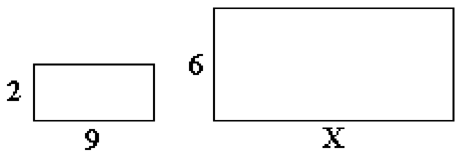
Similar figures have the same shape, but do not have to have the same size. Similar polygons are polygons in which the corresponding angles have the same measure and the ratios of the corresponding sides are the same (or the corresponding sides are proportional). All circles are similar because they have the same shape, but can differ in size depending upon the length of the radii.

The following examples show pairs of similar figures.



Note that in the triangle example, the ratio of the lengths of the altitudes is shown to have the same ratio as that of the corresponding sides. (An altitude of a triangle is a segment drawn from the vertex of a triangle such that it forms a 90? angle with the opposite side.)

Example 1: The following figures are similar. What is the value of x?



$$\frac{2}{6} = \frac{9}{X}$$

Step 1: Determine the ratios of corresponding sides and set them equal to each other to create a proportion to solve for x. In this example, the left ratio compares the widths of the rectangles and the right ratio compares the lengths of the rectangles.

(2)

$$2x = 54$$

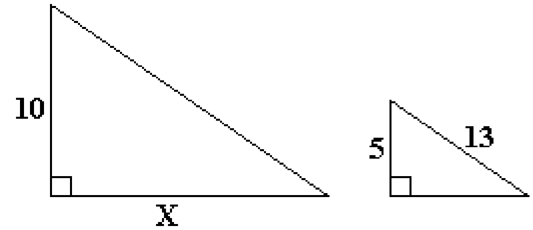
$$\frac{2x}{2} = \frac{54}{2}$$

$$x = 27$$

Step 2: Solve the proportion by cross multiplication. Multiply the numerator of the left fraction by the denominator of the right fraction to get 2x and then multiply the numerator of the right fraction by the denominator of the left fraction to get (6)(9) = 54. Solve by dividing both sides of the equation by 2 to get x = 27.

The value of x is 27.

Example 2: The following figures are similar. What is the value of x?



Note that in this problem, the given sides of the right triangles do not all correspond. The sides of lengths 5 and 10 correspond because they are the shorter legs of each of the given right triangles, but the side of length x is the longer leg of the triangle on the left and the side of length 13 is the hypotenuse of the triangle on the right. The Pythagorean Theorem needs to be applied to solve for the unknown longer leg. The Pythagorean Theorem is given by the following equation:

$$a^2 + b^2 = c^2$$

where a and b are the legs of the right triangle and c is the hypotenuse. Note that it does not matter if a is considered to be the shorter or longer leg.

(1)

$$a^2 + b^2 = c^2$$

$$5^2 + b^2 = 13^2$$

(2)

$$25 + b^2 = 169$$

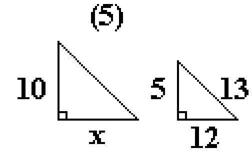
(3)

$$25 + b^2 =$$

$$-25$$

$$b^2 =$$

(5)



(6)

$$\frac{5}{10} = \frac{12}{x}$$

$$5x$$

$$\frac{5x}{5}$$

$$x$$

Step 1: Apply the Pythagorean Theorem to find the longer leg. Substitute 5 in place of a and 13 in place of c.

Step 2: Simplify the terms in the equation.

$5^2 = (5)(5) = 25$

$13^2 = (13)(13) = 169$

Step 3: Subtract 25 from both sides of the equation.

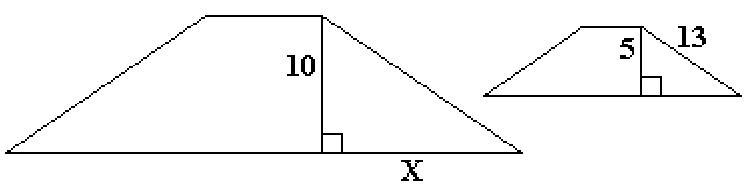
Step 4: Take the square root of both sides and simplify.

Step 5: Use b = 12 because a side of a triangle must be positive and label the diagram with the new information.

Step 6: Determine the ratios of corresponding sides and set them equal to each other to create a proportion to solve for x. Each ratio in this proportion compares corresponding segments from the larger triangle on the left with the smaller triangle on the right. The ratio on the left compares the two shorter legs and the ratio on the right compares the two longer legs.
Step 7: Solve the problem by cross multiplication. Multiply the numerator of the left fraction by the denominator of the right fraction to get 5x and then multiply the numerator of the right fraction by the denominator of the left fraction to get (12)(10) = 120. Solve by dividing both sides of the equation by 5 to get x = 24.

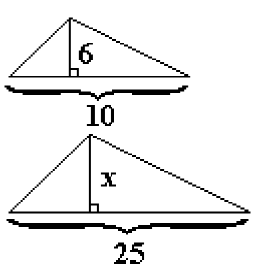
The value of x is 24.

Note that if the right triangles had been positioned in a diagram such as the one below in which the trapezoids are similar, the problem would be completed by the same method as presented in Example 2. The ratio of the corresponding altitudes of similar trapezoids equals the ratio of any pair of corresponding sides.



Example 3: A triangle has a base equal to 10 and height equal to 6. A similar triangle has a base equal to 25. What is the height of the similar triangle?

(1)



(2)

$$\frac{10}{25} = \frac{6}{x}$$

(3)

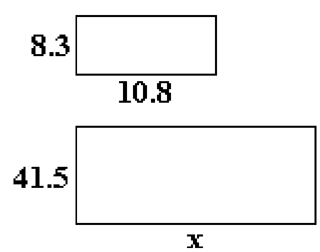
$$10x = 150$$
$$\frac{10x}{10} = \frac{150}{10}$$
$$x = 15$$

Step 1: It is often helpful to draw a diagram before solving the problem. When marking the diagram, label any side to be the base of length 10.The height of length 6 of the triangle is the altitude drawn to the base of length 10. In the similar triangle, let x represent the length of the unknown height which is the altitude drawn to the base of length 25.
Step 2: Determine the ratios of corresponding sides and set them equal to each other to create a proportion to solve for x. Each ratio in this proportion compares corresponding segments from the smaller triangle with the larger triangle. The ratio on the left compares the smaller base to longer base and the right fraction compares the smaller height to longer height.
Step 3: Solve the proportion by cross multiplication. Multiply the numerator of the left fraction by the denominator of the right fraction to get 10x and then multiply the numerator of the right fraction and the denominator of the left fraction to get (6)(25) = 150. Solve by dividing both sides of the equation by 10 to get x = 15.

The height of the similar triangle is 15.

Example 4: A rectangle has length equal to 10.8 and width equal to 8.3. A similar rectangle has width equal to 41.5. What is the length of the similar rectangle?

(1)



(2)

$$\frac{8.3}{41.5} = \frac{10.8}{x}$$

8.

Step 1: It is often helpful to draw a diagram before solving the problem. When marking the diagram, label the length and width of

the smaller rectangle to be 10.8 and 8.3. Label the width of the larger rectangle to be 41.5 and label the length of the rectangle to be x.
Step 2: Determine the ratios of corresponding sides and set them equal to each other to create a proportion to solve for x. Each ratio in this proportion compares corresponding segments from the smaller rectangle with the larger rectangle. The ratio on the left compares the smaller width to the longer width and the right fraction compares the smaller length to longer length.
Step 3: Solve the proportion by cross multiplication. Multiply the numerator of the left fraction by the denominator of the right fraction to get 8.3x and then multiply the numerator of the right fraction and the denominator of the left fraction to get (10.8)(41.5) = 448.2. Solve by dividing both sides of the equation by 8.3 to get x = 54.

The length of the similar rectangle is 54.

Binomial Expansion

A polynomial is the sum of one or more monomials. A monomial is an algebraic expression that has exactly one term. A binomial is a polynomial that contains exactly two terms. An example of a binomial is (a + b). In this example, a is considered one term and b is considered another term. Binomial expansion is the result of rewriting the power of a binomial as a polynomial.

One example of binomial expansion is $(a+b)^2 = a^2 + 2ab + b^2$.

There are two ways to expand binomials. Those two methods are: using the Binomial Theorem and using Pascal's Triangle.

Binomial Expansion Using the Binomial Theorem:

The Binomial Theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

• a and b represent the individual terms of the binomial (such as x and 4)
• n represents the exponent on the original binomial
• r represents the exponent on the "b" term
•
Σ (the Greek letter "sigma") means "the sum c

The other formula that is needed to use the Binomial Theorem is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

In this formula, the n and r represent the same pieces of information they represented in the Binomial Theorem. The "!" represents a factorial. An example of a factorial is 5! = 5 x 4 x 3 x 2 x 1 = 120.

Example 1: Use the Binomial Theorem to expand the binomial.

$$(a+b)^4$$

First, fill in powers of a and b. The powers of a decrease while the powers of b increase as the terms are written down.

$$(a+b)^4 = \_\_\ a^4 + \_\_\ a^3b + \_\_\ a^2b^2 + \_\_\ ab^3 +$$

Second, put in the coefficients. A coefficient is the number that is multiplied by the variables in a term. For example, 2x + 3y, 2 is the coefficient of x and 3 is the coefficient of y.

$$(a+b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b + \binom{4}{2}a^2b^2 + \binom{4}{3}ab$$

There are a few helpful hints for this step in the process. First, in the computation of the coefficients, the n value always stays the same, but the r value will increase by one as you read from the left to the right. Second, the exponent on the b term increases by one as you read from the left to the right and ends when it reaches the value of n. Third,

the exponent on the a term decreases by one as you read from the left to the right and on the final term of the expansion, the exponent of a is zero.

Finally, evaluate the coefficients using $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

The coefficient for $\binom{n}{0}$ will always equal 1.

$$\overset{(1)}{\binom{4}{1} = \frac{4!}{1!(4-1)}} \qquad \overset{(2)}{\binom{4}{1} = \frac{4\cdot3\cdot2\cdot1}{1(3)!}}$$

$$\overset{(3)}{\binom{4}{1} = \frac{4\cdot3\cdot2\cdot1}{1\cdot3\cdot2\cdot1}} \qquad \overset{(4)}{\binom{4}{1} = \frac{24}{6} = 4}$$

Step 1:

Evaluate the coefficient $\binom{4}{1}$ by substituting 4 in place of n and 1 in place of r.

Step 2: Looking at the numerator of the fraction, 4! = 4 x 3 x 2 x 1. We need to evaluate (4 - 1)! in the denominator of the fraction before we can move on.

Step 3: The numerator of the fraction remains the same as in Step 2. If we concentrate on the denominator, 3! = 3 x 2 x 1.

Step 4: There are two ways to evaluate $\binom{4}{1} = \frac{4\cdot3\cdot2\cdot1}{1\cdot3\cdot2\cdot1}$. One way is to multiply to get one number (24) in the numerator and one number (6) in the denominator. Then divide the numerator (24) by the denominator (6) to get 4.

• The other way to evaluate the equation is to "cancel out" any numbers that are the same in the numerator and the denominator.

$$\binom{4}{1} = \frac{4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{1\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

We will still get 4 as the coefficient.

Continue determining the coefficients for all terms in the binomial expansion in the same manner.

Answer:

$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

Example 2: Use the Binomial Theorem to expand the binomial.

$$(5x - 2y)^3$$

First, fill in powers of a and b, using a = 5x and b = -2y.

$$(5x-2y)^3 = \_\_(5x)^3 + \_\_(5x)^2(-2y) + \_\_(5x)(-2y)$$

Second, put in the coefficients.

$$(5x-2y)^3 = \binom{3}{0}(5x)^3 + \binom{3}{1}(5x)^2(-2y) + \binom{3}{2}($$

Third, evaluate the coefficients using $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

$$(5x-2y)^3 = 1(5x)^3 + 3(5x)^2(-2y) + 3(5x)(-2y)^2 +$$

Fourth, evaluate all exponents. Remember that $(5x)^3 = (5x)(5x)(5x) = 125x^3$.

$$(5x-2y)^3 = (1\cdot125x^3) + (3\cdot25x^2\cdot-2y) + (3\cdot5x\cdot$$

Finally, multiply to determine the binomial expansion of $(5x - 2y)^3$

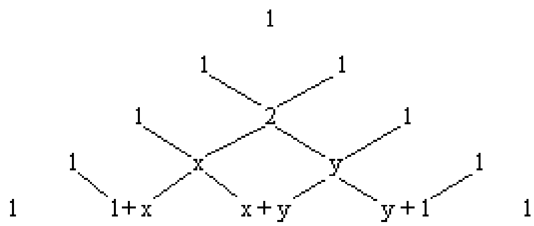
Answer:

$(5x-2y)^3 = 125x^3 - 150x^2y + 60xy^2 - 8y^3$

Binomial Expansion Using Pascal's Triangle:

Pascal's triangle can be thought of as a two-dimensional sequence. Each term is determined by a row and its position in that row. The only term in the top row (row 0) is 1. The first and last elements in all other rows are also 1. If x and y are located next to

each other on a row, the element just below and directly in between them is x + y. See the example below.



Pascal's Triangle

Row 0	1
Row 1	1 1
Row 2	1 2 1
Row 3	1 3 3 1
Row 4	1 4 6 4 1
Row 5	1 5 10 10 5 1
Row 6	1 6 15 20 15 6 1
Row 7	1 7 21 35 35 21 7 1
Row 8	1 8 28 56 70 56 28 8 1
Row 9	1 9 36 84 126 126 84 36 9 1
	⋮

(dots indicate the array goes on without end)

Pascal's triangle helps us with binomial expansion by giving us the coefficients of each term of the binomial expansion. The number of the row of Pascal's triangle corresponds to the exponent of the binomial and the entries in that row of the triangle are the coefficients of the terms in the binomial expansion. The table below illustrates how Pascal's triangle corresponds to binomial expansions.

Powers of (a + b)	Coefficients of the Binomial Expansion	Row of Pascal's Triangle
$(a+b)^0$	1	0
$(a+b)^1$	1 1	1
$(a+b)^2$	1 2 1	2
$(a+b)^3$	1 3 3 1	3
$(a+b)^4$	1 4 6 4 1	4
⋮	⋮	⋮

Example 3: Use Pascal's Triangle and the Binomial Theorem to expand the binomial.

$$(2x+4)^4$$

First, fill in the powers of a and b, using a = 2x and b = 4.

$$(2x+4)^4 = \_\_(2x)^4 + \_\_(2x)^3(4) + \_\_(2x)^2(4)^2 + \_\_(2x)(4$$

Second, put in the coefficients.

$$(2x+4)^4 = \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(4) + \binom{4}{2}(2x)^2(4)^2 + \binom{4}{3}(2x)(4)^3 + \binom{4}{4}(4)^4$$

Third, use Pascal's Triangle to determine the coefficients. Since the original binomial that we want to expand has an exponent of 4, we want to look at the 4th row of Pascal's Triangle for the values of the coefficients. The elements in the fourth row of Pascal's Triangle are: 1 4 6 4 1, so

$$\binom{4}{0}=1, \binom{4}{1}=4, \binom{4}{2}=6, \binom{4}{3}=4, \text{and} \binom{4}{4}=1.$$

$$(2x+4)^4 = 1\cdot(2x)^4 + 4(2x)^3(4) + 6(2x)^2(4)^2 + 4(2x)(4)^3 + 1(4)^4$$

Fourth, evaluate all exponents.

$$(2x+4)^4 = (1\cdot 16x^4) + (4\cdot 8x^3\cdot 4) + (6\cdot 4x^2\cdot 16) +$$

Fifth, multiply to determine the binomial expansion of $(2x+4)^4$

Answer:

$$(2x+4)^4 = 16x^4 + 128x^3 + 384x^2 + 512x + 25$$

Finding the Coefficient of One Term in a Binomial Expansion:

It is possible to use the Binomial Theorem and Pascal's Triangle to determine the coefficient of a single term in a binomial expansion.

Example 4:

Find the coefficient of x^5 in the expansion of

First, we only need the coefficient of one term, so we do not need to fill in the entire expansion. We only need the portion of the expansion where the x has an exponent of 5. We can determine which term we need to focus on by remembering that the sum of the exponents of each term of this binomial should equal 7 (the exponent of the binomial we are expanding). The exponent of the 6 must be 2 (5 + 2 = 7 or 7 - 2 = 5), since we want the x^5 term.

The portion of the expansion we are dealing with is:

$$\_\_\_(x)^5(6)^2$$

Second, we need to determine the coefficient of the term with $(x)^5(\ )^2$. That coefficient would be $\binom{7}{2}$. We now know that the portion of the expansion we are dealing with is:

$$\binom{7}{2}(x)^5(6)^2 .$$

Third,

The 7 in $\binom{7}{2}$ tells us to look in the 7th row of

We need to find the 3rd element (reading from left to right) of the 7th row of Pascal's Triangle because

$$\binom{7}{2}(x)^5(6)^2 .$$

would be the third term of the binomial expansion. The third element of the seventh row of the triangle is 21. The portion of the expansion should now be:

$$21(x)^5(6)^2$$

Finally, we must evaluate the powers and multiply to determine the coefficient of the x^5 term. Six squared is 36. Thirty-six times 21 is 756. The term is now $756x^5$.
Answer: 756

Example 5:

Use Pascal's Triangle to find the binomial coefficient for $\binom{6}{3}$

We refer back to Pascal's Triangle.
$\binom{6}{3}$ is in the form $\binom{n}{r}$. The variable n represents the row in Pascal's Triangle we need to look at. The expression (r + 1) represents the term (reading from left to right) we need to find in row n.

So, to determine the binomial coefficient, we look at the 6th row, 4th term of Pascal's Triangle. That term is 20.

Answer: 20

Evaluating Solutions - B
The Evaluating Solutions skill asks students to determine the reasonableness of possible problem solutions.

Determining the area of a rectangle and a square:

The area of a rectangle is given by the following formula:

$$A = l \cdot w$$

where l represents the length of the rectangle and w represents the width of the rectangle. The unit of measure for area is the square of the units given for the length and width. For example, if the length of a rectangle is 8 inches and the width is 6 inches, then the area would be

$$A = l \cdot w = 8 \cdot 6 = 48$$

The area of the given rectangle is 48 square inches.

A square is a special type of rectangle in which the length and the width are congruent. Thus, the area of a square is given by the following formula:

$$A = s^2$$

where "s" is the length of a side of the square. The unit of measure for area is the square of the units given for the length of a side. If the length of a square is 5 feet, then the area of the square would be

$$A = s^2 = 5^2 = 5 \cdot 5 = 25$$

The area of the given square is 25 square feet.

Determining the length of a side of a square, given its area:

When finding the length of a side of a square, given its area, find the square root of a number, it may be necessary to use the calculator to so. The square root key is indicated by the symbol $\sqrt{}$. On some calculators it is necessary to enter the number first and then the square root key to obtain the answer. On other calculators you may need to enter the square root key first, followed by the number key. It may be necessary to press SHIFT, 2nd, or INV before the $\sqrt{}$.
Example 1: The area of a square is 81 square feet. The length of a side is x inches. What is the length of each side?

(1) $A = s^2$ **(2)** $81 = s^2$ **(3)** $\pm\sqrt{81} = \sqrt{s^2}$, $\pm 9 = s$ **(4)** **The length of each side is 9 inches.**

Step 1: Write down the correct area formula.
Step 2: Substitute 81 in place of A.
Step 3: Take the square root of both sides. The square roots of 81 are +9 and - 9.
Step 4: Only the positive square root can be used when solving the word problem. (Negative numbers can not be used to represent the length of sides of a geometric figure.)

Determining the length or width of a rectangle, given its area:

Example 2: The area of a rectangle is 24 square inches. The length is (x + 3) inches and the width is
(x - 2) inches. What is the length of the rectangle?

(1) $A = l \cdot w$ **(2)** $24 = (x+3)(x-2)$ **(3)** $24 = x^2 + x - 6$ **(4)**

$$\begin{array}{r} 24 = x^2 + x - 6 \\ -24 \qquad\qquad -24 \\ \hline 0 = x^2 + x - 30 \end{array}$$

(5) $0 = (x+6)(x-5)$ **(6)**

$$\begin{array}{r} (x+6) = 0 \\ -6 \quad -6 \\ \hline x = -6 \end{array} \quad \text{or} \quad \begin{array}{r} (x-5) = 0 \\ +5 \quad +5 \\ \hline x = 5 \end{array}$$

(7) **The length of the rectangle is** $x + 3 = 5 + 3 = 8$ **inches.**

Step 1: Write down the correct area formula.
Step 2: Substitute the given values and expressions into the formula. A = 24, l = (x + 3) and w = (x - 2)
Step 3: Multiply the binomials using the distributive property or the **FOIL** method and simplify. (Recall that in the **FOIL** method, multiply the **F**irst elements in each factor of the binomial to get (x)(x) = x^2. Next multiply the pairs of **O**uter and **I**nner elements to get (3)(x) = 3x and (x)(-2) = -2x, respectively. Combine the like terms 3x + (-2x) to get x. Then multiply the **L**ast elements to get (3)(-2) = -6.)
Step 4: Add -24 to both sides of the equation to get 0 on one side of the equal sign.
Step 5: Factor the trinomial. Recall that to factor a trinomial, undo the process of the FOIL method.
• The "x" in each binomial factor came from $x^2 = (x)(x)$. • To get the other elements of the binomial factors, look for two numbers which when multiplied will give -30, but when added will equal 1, the coefficient of the x term in the trinomial $x^2 + x - 30$. The two numbers that satisfy those two conditions are 6 and -5.
Step 6: Set each of the factors equal to zero, so x + 6 = 0 or x - 5 = 0. Solve to get the two answers, -6 and 5.
Step 7: Only the positive value for x can be used when solving the word problem. The expression (x + 3) represents the length of the rectangle, thus substitute 5 in place of the x and simplify.

Answer: 8 inches

If the above example had asked for the width of the rectangle, then Step 7 would have been written as follows:

(7) The width of the rectangle is (x - 2) = 5 - 2 = 3 inches.

Step 7: Only the positive value for x can be used when solving the word problem. The expression (x - 2) represents the length of the rectangle, thus substitute x by 5 and simplify.

Example 3: A rectangle has a width that is 3 more inches than its length. The area of the rectangle is 70 square inches. What is the width of the rectangle?

(1)

Let x = length of rectangle
Let x+3 = width of rectangle

(2)

$A = l \cdot w$

(3)

$70 = (x)(x + 3)$

(4)

$70 = x^2 + 3x$

(5)

$$\begin{array}{rl} 70 = & x^2 + 3x \\ -70 & \qquad -70 \\ \hline 0 = & x^2 + 3x - 70 \end{array}$$

(6)

$0 = (x + 10)(x - 7)$

(7)

$$\begin{array}{rcl} x + 10 = 0 & \text{or} & x - 7 = 0 \\ -10 \;\; -10 & & +7 \;\; +7 \\ \hline x = -10 & \text{or} & x = 7 \end{array}$$

(8)

The width of the rectangle is $x + 3 = 7 + 3 = 10$ inches.

Step 1: Set up the correct "Let" statement to identify the unknown values in the problem. It is easier to let x represent the smaller amount (length) and since the width is 3 more inches than the length, the width is represented by x + 3.
Step 2: Write down the correct area formula.
Step 3: Substitute the given values and expressions into the formula. A = 70, l = (x) and w = (x + 3).
Step 4: Multiply using the distributive property.
Multiply $(x)(x) = x^2$ and $(x)(3) = 3x$ Step 5: Add -70 to both sides of the equation to get 0 on one side of the equal sign.
Step 6: Factor the trinomial. (Refer to Step 5 in he previous example, if needed.)
Step 7: Set each of the factors equal to 0 and solve to get the two answers, -10 and 7.
Step 8: Only the positive value for x can be used when solving the word problem. The expression (x + 3) represents the width of the rectangle, thus substitute 7 in place of x and simplify.

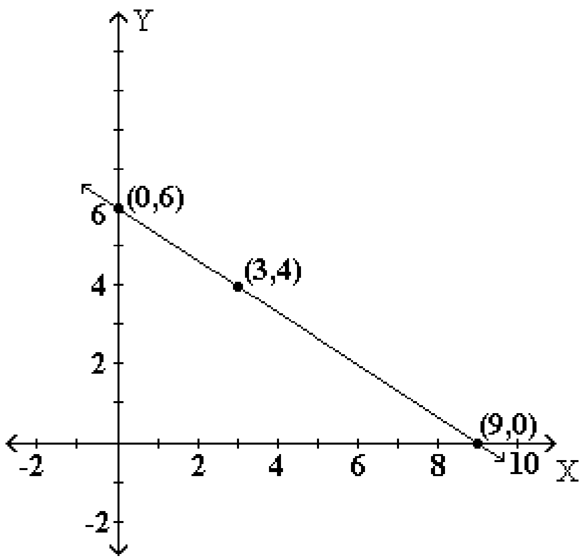
Maximizing or minimizing a function given specific constraints:

The process of maximizing or minimizing a given function, sometimes called an objective quantity, is subject to specific constraints when used in different application problems. The constraints are often expressed as linear inequalities and one method used to solve this type of problem is called linear programming.

To solve this type of problem, it is necessary to graph the given constraints or inequalities. Two constraints that are commonly given are $x \geq 0$ and $y \geq 0$. These constraints limit the graphs of the remaining inequalities to lie only in the first quadrant (where the x and y values are positive as well as the positive x-axis and y-axis values).

Next, graph the other linear inequalities. Recall that to graph an inequality, first sketch a graph of the corresponding equality to get the boundary line for the resulting shaded region. For example, first sketch the corresponding equality 2x + 3y = 18 to graph $2x + 3y \geq 18$ Recall that to sketch 2x + 3y = 18, it is helpful to use an x/y chart (table of values) to find three ordered pairs that lie on the graph of the linear equation. Then plot the three points and draw the line that contains them to get the graph of 2x + 3y = 18.

x	y
0	6
9	0
3	4



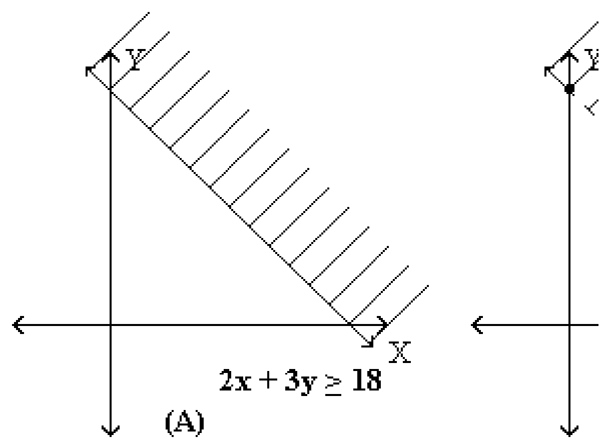
This line divides the rectangular coordinate system into two half-planes. The graph of the solution to an inequality is a shaded region, or half-plane, either above or below the line. To determine which region is correct, pick a test point that does not lie on the graphed line, (often (0, 0) is used), and substitute this point into the inequality to determine if it makes the inequality a true or false statement.

$$2x + 3y \geq 18$$

$$2(0) + 3(0) \geq 18 \quad \text{Substitute } x = 0 \text{ and } y = 0$$

$$0 \geq 18 \quad \text{False statement}$$

The point (0, 0) does not lie in the half-plane that represents the solution set to the linear inequality because 0 is NOT greater than 18. Thus the other half-plane is shaded to represent the correct graph to the inequality as shown below by graph A.



The graph of the corresponding equality would have been a dotted line (not a solid one) if the given linear inequality had been $2x + 3y > 18$ see graph B above.

Example 4: What is the minimum value given the following constraints?

Objective Quantity:
$C = 4x + 5y$

Constraints:

$x \geq 0$
$y \geq 0$
$2x + 3y \geq 18$
$2x + y \geq 10$

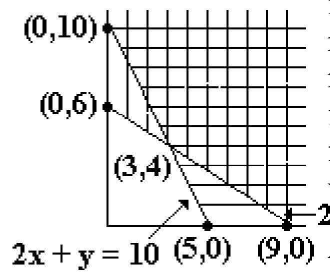
(1)

$$\begin{array}{r} 2x + 3y = 18 \\ 2x + y = 10 \\ \hline 2y = 8 \\ y = 4 \end{array}$$

(2)

$$\begin{array}{r} 2x + 3y = 18 \\ 2x + 3(4) = 18 \\ 2x + 12 = 18 \\ \hline 2x = 6 \\ x = 3 \end{array}$$

(3)



(4)

The Vertices are:
(0,10), (3,4) and (9,0)

(5)

$C = 4x + 5y$
$(0,10)\ C = 4(0) + 5(10) = 50$
$(3,4)\ C = 4(3) + 5(4) = 52$
$(9,0)\ C = 4(9) + 5(0) = 36$

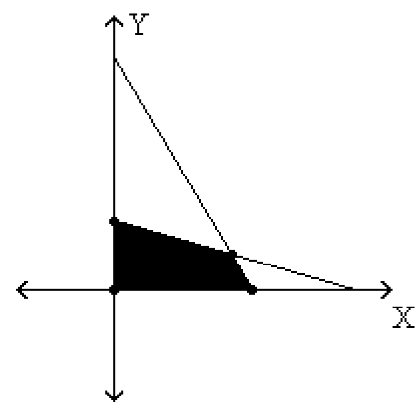
(6)

The minimum Value is 36.

Step 1: Solve the system of corresponding equalities to find the point of intersection of the boundary lines. First subtract the bottom equation from the top one (or you could multiply the bottom equation by (-1) and then add the two equations). Then solve the resulting equation 2y = 8 to get y = 4.
Step 2: Substitute y = 4 into the top equation to find the corresponding x-value and get x = 3. The intersection point is (3, 4).
Step 3: Graph the four linear inequalities. First sketch the boundary lines 2x + 3y = 18 and 2x + y = 10. Then shade the correct half-planes. Label the boundary lines and the point of intersection found in Steps 1 and 2.
Step 4: Locate the vertices of the region determined by the overlapping shaded regions found in the completed graph of Step 3.
Step 5: Evaluate the Objective Quantity for each of the three vertices found in Step 3 by substituting each of the x and y values into the expression 4x + 5y. Then simplify.
Step 6: Choose the smallest number in the list of values from Step 5 to find the minimum value for the Objective Quantity function.

The minimum value is 36.

Note that the problem could have asked to determine the maximum value for the Objective Quantity. In the problem of Example 1, no maximum value could be found because the overlapping shaded region found in Step 3 is not a bounded region, that is, the overlapping region is not fully contained in a closed region such as the figure below. Thus there is no maximum value that can be found for C = 4x + 5y. If the region had been bounded, then each of the vertices of the bounded region would be substituted in the Objective Quantity, as shown by Step 5 above, to determine the maximum value. The largest number found would then give the maximum value.



Irrational Numbers: Pi
An irrational number is a number that cannot be written as a fraction. The decimal equivalents of irrational numbers do not terminate (end) and never repeat. For example, 0.10110111011110...and $\pi \sim 3.14159265...$ are two decimals that never repeat and never end.

Determining the Circumference of a Circle:

Circumference is the measurement of the

distance around a circle. The circumference of a circle is found by using the following formula:

$$C = 2\pi r$$

In this formula, r represents the radius of the circle. The radius of a circle is the length of the segment from the center of a circle to any point on the circle. The radius is one-half the length of the diameter (the distance across a circle) of a circle. Pi is the ratio of the circumference of any circle to its diameter and is approximately equal to 3.14159. Pi is represented by the symbol π. There is a key on the calculator that represents π. It may be necessary to press SHIFT, 2nd, or INV before the π key. If a calculator does not have a π key, it is acceptable to approximate π with 3.14.

Example 1: Find the circumference of a circle with a diameter of 18 inches.

(1) $r = 18 \div 2$

(2) $C = 2\pi \cdot 9$

(3) $C = 18\pi$

Step 1: Since the radius of a circle is one-half the diameter, divide 18 by 2 to get the value of the radius(9).
Step 2: Substitute 9 in place of r in the formula for the circumference of a circle.
Step 3: Multiply 9 by 2 (all three terms are multiplied and it does not matter which order numbers are multiplied).

The circumference is 18pi inches.

Example 2: Find the diameter of a circle whose circumference is 87 feet.

(1)	(2)	(3)
$87 = 2\pi r$	$\frac{87}{2\pi} = \frac{2\pi r}{2\pi}$	$\frac{2 \cdot 87}{2\pi}$
	$\frac{87}{2\pi} = r$	

Step 1: Substitute 87 in place of C because 87 is the circumference.
Step 2: To isolate the r on one side of the equation, divide each side of the equation by 2π Step 3: Since the radius is half the length of the diameter, multiply the radius by 2 to get the length of the diameter. Remember, what is done to one side of an equation must be done to the other side in order for the equation to balance, so multiply the left side of the equation by 2 also. The 2 in the numerator cancels with the 2 in the denominator on the left side of the equation.
Step 4: The diameter of the circle equals 87/pi feet.

Finding the Area of a Circle:

The area of a figure is the measure, in square units, of the interior region of a two-dimensional figure. The area of a circle is found by using the following formula:

$$A = \pi r^2$$

In this formula, r represents the radius of the circle.

Example 3: Find the area of the circle whose radius is 10.2 inches.

(1)	(2)
$A = \pi(10.2)^2$	$A = 104.04\pi$

Step 1: Substitute 10.2 in place of r because the radius of the circle is 10.2.
Step 2: Square 10.2 (10.2 x 10.2) and multiply the answer by π.

The area of the circle whose radius is 10.2 is

104.04pi square inches.

Example 4: Find the area of the circle whose circumference is 35 inches.

(1)

$C = 2\pi r$

$35 = 2\pi r$

(2)

$\frac{35}{2\pi} = \frac{2\pi r}{2\pi}$

(3)

$r = \frac{17.5}{\pi}$

(4)

$A = \pi r^2$

$A = \pi \left(\frac{17.5}{\pi}\right)^2$

(6)

$A = \frac{\pi}{1} \cdot \frac{306.25}{\pi^2}$

(7)

$A = \frac{306.25}{\pi}$

Step 1: Before the area of the circle can be determined, the radius of the circle must be calculated. Since the circumference is known, the formula for the circumference of a circle can be used to determine the radius of the circle. Substitute 35 in place of C in the circumference formula.

Step 2: To isolate the r on one side of the equal sign, divide each side of the equation by 2π. Step 3: Divide 35 by 2 and the radius is $\frac{17.5}{\pi}$. Step 4: Now that the radius of the circle is known, it can be substituted in place of r in the formula for the area of a circle

Step 5: $\left(\frac{17.5}{\pi}\right)^2$ can be rewritten as $\frac{(17.5)^2}{(\pi)^2}$. Step 6: Square 17.5 (17.5 x 17.5) to get 306.25.

Rewrite π as a fraction $\left(\frac{\pi}{1}\right)$. Step 7: The pi in the numerator and one pi in the denominator divide out and leave one pi in the denominator.

The area of the circle, whose circumference is 35 inches, is 306.25/pi square inches.

Finding the Area of an Ellipse:

An ellipse is a closed two-dimensional plane figure that is oval in shape. Every ellipse has two axes. The two axes lie on the symmetry lines and intersect at the center O of the ellipse. One of the axes is the major axis. The major axis contains the foci, has two vertices of the ellipse as its endpoints, and is always longer.

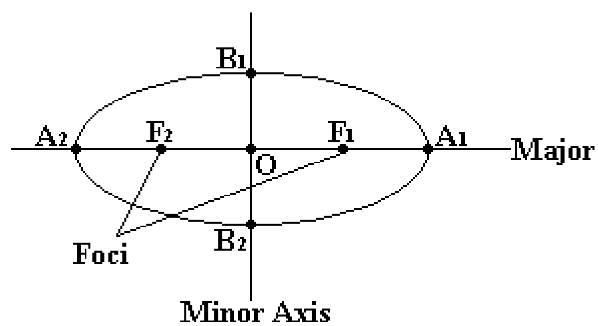
(5)

The length of the major axis is the distance fr

$A = \pi \cdot \frac{(17.5)^2}{\pi^2}$

The other axis of an ellipse is called the minor axis. The minor axis does not contain the foci and has two co-vertices of the ellipse as its endpoints.

The length of the minor axis is the distance fr



The formula for an ellipse with axes of lengths 2a and 2b follows.

$$A = \pi ab$$

The length of the major axis is 2a and the length of the minor axis is 2b.

Example 5: Find the area of an ellipse

whose minor axis is 23 feet and major axis is 35 feet.

$$(1)\quad 2a = 35 \quad 2b = 23 \qquad (2)\quad \frac{2a}{2} = \frac{35}{2} \quad \frac{2b}{2} = \frac{23}{2} \qquad (3)\quad a = 17.5 \quad b = 11.5$$

$$(4)\quad A = \pi \cdot 17.5 \cdot 11.5 \qquad (5)\quad A = 201.25\pi$$

<u>Step 1</u>: The lengths of the axes can be used to determine a and b, so the formula for the area of an ellipse can be used. Since the major axis is 35 feet long, 2a = 35. The minor axis is 23 feet long, so 2b = 23.
<u>Step 2</u>: Divide each side of each equation by 2 to isolate the variable in each equation.
<u>Step 3</u>: Divide 35 by 2 to determine that a = 17.5. Divide 23 by 2 to determine that b = 11.5.
<u>Step 4</u>: Substitute the values of a and b into the formula for the area of an ellipse.
<u>Step 5</u>: Multiply 17.5 by 11.5 (numbers can be multiplied in any order) and multiply that product by π.
The area of the ellipse is A = 201.25pi square feet.

Example 6: Find the minor axes of an ellipse whose major axis is 25 inches and whose area is 55 square inches.

$$(1)\quad \frac{2a}{2} = \frac{25}{2},\ a = 12.5 \qquad (2)\quad 55 = \pi \cdot 12.5 \cdot b \qquad (3)\quad \frac{55}{\pi \cdot 12.5} = \frac{\pi \cdot 12.5 \cdot b}{\pi \cdot 12.5}$$

$$(4)\quad \frac{4.4}{\pi} = b \qquad (5)\quad 2 \cdot \left(\frac{4.4}{\pi}\right) = b \cdot 2 \qquad (6)\quad 2b = \frac{8.8}{\pi}$$

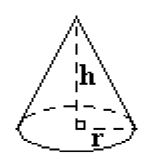
<u>Step 1</u>: Since the area of the ellipse and the length of the major axis are known, the formula for the area of an ellipse can be used to determine the length of the minor axis. The length of the major axis (25) can be used to find the value of a, so 2a = 25. Divide each side of the equation by 2 to isolate the variable.
25 ÷ 2 = 12.5, so a = 12.5.
<u>Step 2</u>: Substitute the value of the area (55) and the value of a (12.5) into the formula for the area of an ellipse. We can use this new equation to solve for b (which is 1/2 the length of the minor axis).
<u>Step 3</u>: To isolate b on one side of the equal sign, divide each side of the equation by 12.5π. <u>Step 4</u>: Divide 55 by 12.5 to determine that b = $\frac{4.4}{\pi}$. <u>Step 5</u>: Since the length of the minor axis is 2b and we know the value of b, we need to multiply each side of the equation by 2 to determine the length of the minor axis.
<u>Step 6</u>: Multiply b by 2 to get 2b.
Multiply $\frac{4.4}{\pi}$ by 2 to get $\frac{8.8}{\pi}$.
The length of the minor axis of the ellipse is 8.8/pi inches.

Finding the Volume of a Cone:

A <u>cone</u> is a 3-dimensional figure with one curved surface, one flat surface (usually circular), one curved edge, and one vertex. The <u>volume</u> is the number of cubic units it takes to fill a figure. The formula for the volume of a cone is:

$$V = \frac{1}{3}\pi r^2 h$$

In this formula, r represents the radius of the circle (base) and h represents the height of the cone. (See the diagram of a cone below.)



Example 7: Find the volume of a cone that has a radius of the base equal to 8 inches and height equal to 12 inches.

$$(1)\quad \frac{1}{3}\pi\left(8^2\right)\left(12\right)$$
$$(2)\quad \frac{1}{3}\pi\left(64\right)\left(12\right)$$
$$(3)\quad 256\pi$$

Step 1: Since the radius r and the height h are known, substitute the values into the formula for the volume of a cone.
Step 2: Following the order of operations, square the 8 first (8 x 8 = 64).
Step 3: Multiply 64 and 12, then multiply that product by 1/3. (64 x 12 = 768; 768 x 1/3 = 256)

The volume of the cone is V = 256pi cubic inches.

Example 8: Find the height of a cone that has a radius of the base equal to 10 inches and volume equal to 820 cubic inches.

(1) $820 = \frac{1}{3}\pi\left(10^2\right)h$

(2) $820 = \frac{1}{3}\pi\left(100\right)h$

(3) $\frac{820}{100\pi} = \frac{\frac{1}{3}\pi\left(100\right)h}{100}$

(4) $\frac{3}{1}\cdot\frac{8.2}{\pi} = \frac{3}{1}\cdot\frac{1}{3}h$

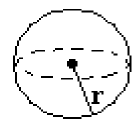
(5) $\frac{24.6}{\pi} = h$

Step 1: Since the volume of the cone and the radius of the base are known, the formula for the volume of a cone can be used to determine the height of the cone. Substitute the value of the volume and the value of the radius into the formula for the volume of a cone.
Step 2: Following the order of operations, square the 10 (10 x 10 = 100).
Step 3: To begin to isolate the h on one side of the equal sign, divide each side of the equation by 100π. Divide 820 by 100 to get 8.2.
Step 4: Multiply each side of the equation by the inverse of 1/3, which is 3/1, to finish isolating the h on one side of the equation. The 3 in the numerator divides out with the 3 in the denominator.
Step 5: Multiply 8.2 by 3 get 24.6.

The height of the cone is 24.6/pi inches.

Finding the Volume of a Sphere:

A sphere is a 3-dimensional figure made up of all points in space that are equally distant from a given point called the center. The best example of a sphere is a ball.



The formula for the volume of a sphere is:

$$V = \frac{4}{3}\pi r^3$$

In this formula, r represents the radius of the sphere.

Example 9: Find the volume of a sphere whose radius is 11 inches.

(1) $V = \frac{4}{3}\pi\left(11\right)^3$

(2) $V = \frac{4}{3}\pi\left(1331\right)$

(3) $V = 1774.67\pi$

Step 1: Substitute the value of the radius (11) into the formula for the volume of a sphere.
Step 2: Following the order of operations, calculate $11^3 = 11\times11\times11 = 1,331$. Step 3: Multiply 1331 by 4/3. To do this, first multiply 1331 by 4, then divide the product by 3 (1331 x 4 = 5324; 5324 ÷ 3 ~ 1774.67).

The volume of the sphere is V = 1.774.67pi cubic inches.

Example 10: Find the radius of the sphere whose volume is 1,432 cubic feet.

(1)

$$1432 = \frac{4}{3}\pi r^3$$

(2)

$$\frac{3}{4} \cdot 1432 = \cancel{\frac{3}{4}} \cdot \cancel{\frac{4}{3}} \pi r^3$$

(3)

$$\frac{1074}{\pi} = \frac{\cancel{\pi} r^3}{\cancel{\pi}}$$

(4)

$$\sqrt[3]{\frac{1074}{\pi}} = \sqrt[3]{r^3}$$

(5)

$$r = \sqrt[3]{\frac{1074}{\pi}}$$

Step 1: Since the volume of the sphere is known (1432), the formula for the volume of a sphere can be used to find the radius of the sphere. Substitute the value of the volume of the sphere into the formula for the volume of a sphere.
Step 2: Multiply both sides of the equation by the reciprocal of 4/3, which is 3/4, to begin isolating the r. To multiply 1432 by 3/4, multiply 1432 by 3 then divide that product by 4 (1432 x 3 = 4296; 4296 ÷ 4 = 1074).
Step 3: Divide both sides of the equation by pi to isolate the r^3. Step 4: Take the cube root of each side of the equation. This will isolate the r on one side of the equal sign.
Step 5: Once the r has been isolated, the equation can be written beginning with the r. It is acceptable to leave the radical sign in the answer.

The radius of the sphere is $r = \sqrt[3]{\frac{1074}{\pi}}$ feet.

Irrational Numbers: e

An irrational number is a number that cannot be written as a fraction. The decimal equivalents of irrational numbers do not terminate (end) and never repeat. For example, 0.10110111011110...and $\pi \sim$ 3.14159265... are two decimals that never repeat and never end.

Continuously Compounded Interest:

Compound interest is the interest paid on the previously earned interest as well as the initial deposit. The initial deposit is called the principal and is denoted with the variable P. The interest rate (denoted with the variable r) is the percent of the principal that will be earned over a particular period of time (usually one year). The variable t represents the number of years (time) the principal is invested.

Many times, interest is compounded once a year (annually), twice a year (semiannually), or four times a year (quarterly). Interest can also be compounded continuously. There is a special formula for compounding interest continuously. That formula is:

$$A = Pe^{rt}.$$

Like the number pi, 'e' looks like a variable, but it is an irrational number which can be expressed as an infinite, non-repeating decimal.

$$e^{0.05}$$

There is a button on the calculator specifically for calculating 'e' to any power. That button is e^x. It may be necessary to press the SHIFT, INV, or 2nd key before the e^x.

Example 1: Calculate.

$$e^{1.23}$$

The calculator key sequence for most scientific calculators is as follows:

Step 1: Enter 1.23 into the calculator.
Step 2: Press the SHIFT, INV, or 2nd key.
Step 3: Press the e^x key.

Answer: $e^{1.23} \approx 3.42$

Some calculators may require pressing the equal sign after the e^x key. If this key

sequence does not work for your calculator, consult the manual that came with the calculator.

Example 2: Suppose that you deposit \$12,000.00 in an account that pays 6.75%, compounded continuously, which is modeled by the formula $A = Pe^{rt}$. If you leave it in the account for 5 years, how much money will you have?

(1) $P = 12,000;\ r = 6.75\% = 0.0675;\ \text{and}\ t = 5$

(2) $A = 12000e^{0.0675(5)}$

(3) $A = 12,000e^{0.3375}$

(4) $A = 12,000 \cdot 1.40139608$

(5) $A = 16817.2753$

Step 1: The principal (initial deposit) is \$12,000, so p = 12,000. The interest rate is 6.75%. The interest rate must be converted from a percent to a decimal, so r = 0.0675. The money will be in the account for 5 years, so t = 5.
Step 2: Substitute the known values in place of the variables that represent them.
Step 3: Multiply 0.0675 by 5 to get the exponent for e. Step 4: Applying the order of operations, $e^{0.3375} = 1.401439608$ (DO NOT ROUND!)
Step 5: Multiply 1.401439608...by 12,000.

After 5 years, you would have \$16,817.28 in the account.

Other Uses of the Irrational Number e:

The irrational number 'e' can also be used to determine population growth, population decay, air pressure, and other types of growth and decay. In these cases, the formula will be given to you as part of the problem.

Example 3: Four hundred grams of radium are stored. The amount of radium present after t years is modeled by the formula below.

$$R = 400e^{(-0.00043)t}.$$

How much radium will be left after 12,000 years?

(1) $t = 12,000$

(2) $R = 400e^{(-0.00043)(12,000)}$

(3) $R = 400e^{(-5.16)}$

(4) $R = 400 \cdot 0.005741699686$

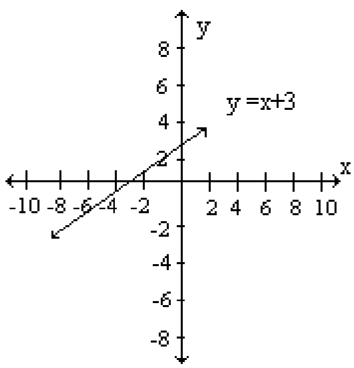
(5) $R = 2.296679874$

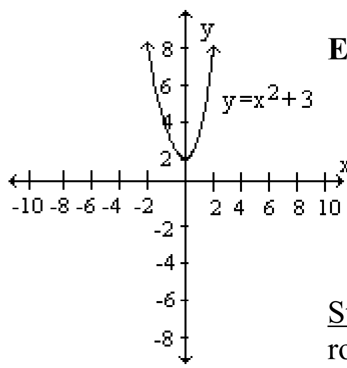
Step 1: Since the radium will be left for 12,000 years, t = 12,000.
Step 2: Substitute 12,000 into the equation in place of t.
Step 3: Multiply -0.00043 by 12,000 to determine the exponent on the e. Step 4: Applying the order of operations, $e^{(-5.16)} = 0.005741699686$ Step 5: (DO NOT ROUND!) Multiply 0.005741699686...by 400.

There will be approximately 2.30 grams of radium left after 12,000 years.

Non-Linear Equations
A non-linear equation is an equation whose graph is **not** a straight line.

An example of a linear equation (graph is a straight line) is y = x + 3 and an example of a non-linear equation is $y = x^2 + 3$. These two equations are graphed below.





A basic rule to follow to determine whether an equation is linear or non-linear is that non-linear equations have variables that have powers other than one and linear equations have powers equal only to one.

Simplifying Square Roots:

Before we can begin solving non-linear equations, we must discuss how to simplify square roots. When a number is multiplied by itself the product is the square of the number. A square root of a number is a factor that when multiplied by itself equals the number. For example: 2 x 2 = 4, so 4 is the square of 2 also, since 4 = 2 x 2, 2 is a square root of 4. Another square root of 4 is -2 because -2 x -2 = 4. Numbers that have a rational number as their square root are called perfect squares. Examples of perfect squares are: 9 (square root is 3), 16 (square root is 4), 25 (square root is 5) and 9/25 (square root is 3/5).

The notation for a square root is this symbol: $\sqrt{}$.

$\sqrt{16}$ is read " the square root of 16" or " radica

To simplify a square root, we first determine two factors that multiply to make the whole number. One of these two factors should be a perfect square, preferably the largest perfect square that is a factor of the number. Then we take the square root of the perfect square factor and place that number in front of the radical symbol.

Example 1: Simplify.

$\sqrt{72}$

(1) $\sqrt{72} = \sqrt{36 \cdot 2}$

(2) $\sqrt{72} = \sqrt{36} \cdot \sqrt{2}$

(3) $\sqrt{72} = 6\sqrt{2}$

Step 1: Rewrite the problem as the square root of two factors of 72. Remember, one of the factors should be the largest perfect square that is a factor of 72. In this case 36 and 2 were used because 36 times 2 equals 72 and 36 is a perfect square.
Step 2: Now the problem can be rewritten as the square root of 36 times the square root of 2.
Step 3: Determine the square root of 36 (which is 6) and multiply it by radical 2. Since 2 does not have a factor that is a perfect square, radical 2 does not change.

Example 2: Another way to simplify $\sqrt{72}$

(1) $\sqrt{72} = \sqrt{9 \cdot 8}$

(2) $\sqrt{72} = \sqrt{9 \cdot 4 \cdot 2}$

(3) $\sqrt{72} = \sqrt{9} \cdot \sqrt{4} \cdot \sqrt{2}$

(4) $\sqrt{72} = 3 \cdot 2 \cdot \sqrt{2}$

(5) $\sqrt{72} = 6\sqrt{2}$

Step 1: It is possible to simplify a square root if the largest perfect square factor is not known. Once again, rewrite the problem as the square root of two factors of 72. Make sure one of these two factors is a perfect

square. In this case 9 and 8 were used because 9 is a perfect square and 9 times 8 equals 72.
Step 2: Since 8 has a factor that is a perfect square (4), the problem must be rewritten as the product of these three factors of 72. 9 x 4 x 2 = 72.
Step 3: Since 9 and 4 are perfect squares and 2 does not have a perfect square factor, the problem can be rewritten as the square root of 9 times the square root of 4 times the square root of 2.
Step 4: Determine the square root of 9 (which is 3), multiply it by the square root of 4 (which is 2), and multiply them both by radical 2.
Step 5: Finally, multiply 3 and 2 to get 6. The 6 is multiplied by radical 2 to obtain the final answer: $6\sqrt{2}$.

Solving Non-Linear Equations:

Non-Linear equations can be solved in much the same way as linear equations. The goal of solving a non-linear equation is to isolate the variable on one side of the equal sign.

Example 3: Solve the following equation.

$$-2x^2+18=3x^2-12$$

$$\begin{array}{lrcl}
 & -2x^2+18 & = & 3x^2-12 \\
(1) & +2x^2 & & +2x^2 \\
\hline
 & 18 & = & 5x^2-12 \\
(2) & +12 & & +12 \\
\hline
(3) & \frac{30}{5} & = & \frac{5x^2}{5} \\
(4) & x^2 & = & 6 \\
(5) & \sqrt{x^2} & = & \sqrt{6} \\
(6) & x & = & \pm\sqrt{6}
\end{array}$$

Step 1: Add $2x^2$ to each side of the equation, p
of the x^2 terms on the same side of the equal s
Step 2: Add 12 to each side of the equation, pl
the terms that do not have an x^2 on the opposi
Step 3: Divide each side of the equation by 5. '
the equal sign.
Step 4: $6 = x^2$ can be rewritten as $x^2 = 6$, beca
which is written first.
Step 5: In order to solve the equation, we must
power of 2 from x^2, we must take the square r
Step 6: All terms have a positive and a negati
front of the answer. Six does not have a perfec
any further.

Answer: $x = \pm\sqrt{6}$

Example 4: Solve the following equation.

$$10x^2+9=8x^2+33$$

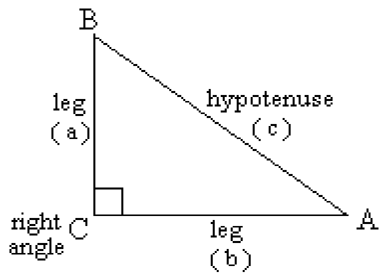
$$\begin{array}{lrcl}
 & 10x^2+9 & = & 8x^2+33 \\
(1) & -8x^2 & & -8x^2 \\
\hline
 & 2x^2+9 & = & 33 \\
(2) & -9 & & -9 \\
\hline
(3) & \frac{2x^2}{2} & = & \frac{24}{2} \\
(4) & x^2 & = & 12 \\
(5) & \sqrt{x^2} & = & \pm\sqrt{12} \\
(6) & x & = & \pm 2\sqrt{3}
\end{array}$$

Step 1: Subtract $8x^2$ from each side of the equ
Step 2: Subtract 9 from each side of the equati
Step 3: Divide each side of the equation by 2 t
Step 4: 24 divided by 2 is 12, so $x^2 = 12$.
Step 5: Take the square root of each side of the
Step 6: Simplify $\sqrt{12}$. Remember that every te
be placed in front of the answer.

Using the Pythagorean Theorem to Solve Non-Linear Equations:

The Pythagorean Theorem can be used to determine the length of a missing side of a right triangle. A right triangle is a triangle that has one right (90º) angle. A 90º angle is marked in a triangle with a box in the angle. The Pythagorean Theorem states that the sum of the squares of the legs of a right triangle is equal to the square of the

hypotenuse. The hypotenuse of a right triangle is the side of the triangle opposite the right angle. The other two sides of the triangle are called the legs.

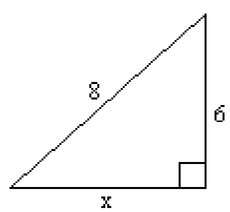


The Pythagorean Theorem states:

$$a^2 + b^2 = c^2$$

In the Pythagorean Theorem, 'c' represents the length of the hypotenuse and 'a' and 'b' represent the lengths of the legs of the right triangle. The Pythagorean Theorem only works for right triangles.

Example 5: Use the Pythagorean Theorem to solve for x.



(1) $c^2 = a^2 + b^2$

$a = x, b = 6, c = 8$

(2) $8^2 = x^2 + 6^2$

(3) $64 = x^2 + 36$

(4) $\begin{aligned} 64 &= x^2 + 36 \\ -36 & \quad\;\; -36 \\ 28 &= x^2 \end{aligned}$

(5) $\sqrt{x^2} = \sqrt{28}$

(6) $\begin{aligned} x &= \sqrt{4 \cdot 7} \\ x &= 2\sqrt{7} \end{aligned}$

Step 1: Write the Pythagorean Theorem. Then determine the values of a, b, and c. Remember, c always represents the length of the hypotenuse. In this triangle, the length of the hypotenuse is 8, so c = 8. It does not matter whether a or b is assigned the value of x or the value of 6.
Step 2: Substitute the values of a, b, and c into the Pythagorean Theorem.
Step 3: Following the order of operations, square the 8 (8 x 8 = 64) and the 6 (6 x 6 = 36).
Step 4: Subtract 36 from each side of the equation.
Step 5: Take the square root of each side of the equation.
Step 6: Simplify radical 28. In this case, the variable represents the length of the side of a triangle, so the answer cannot be negative.

Answer: $2\sqrt{7}$.

Solving an Equation by Completing the Square:

Completing the square is a method of

solving a quadratic equation in order to express the equation as a single squared term. The method of completing the square is used when an equation cannot be factored. Completing the square involves adding the square of one term to the equation and solving the equation for the value of the variable. The theorem for completing the square states:

To complete the square on $x^2+bx=c$, add $(\frac{1}{2}b)^2$

the result will be $x^2+bx+\left(\frac{1}{2}b\right)^2=c+\left(\frac{1}{2}b^2\right)$ such that

$$\left(x+\frac{1}{2}b\right)^2=c+\left(\frac{1}{2}b\right)^2$$

The following is a detailed example of how to complete the square.

Example 6: Solve the equation by completing the square.

$$x^2+8x-7=0$$

(1) $x^2+8x-7=0$

$$\begin{array}{r} x^2+8x-7=0 \\ +7 \quad +7 \\ \hline x^2+8x=7 \end{array}$$

(2) $b=8$

$$x^2+8x+\left(\frac{1}{2}\cdot 8\right)^2=$$

$$x^2+8x+4^2=7+4$$

(3) $(x+4)^2=7+16$

(3) $\sqrt{(x+4)^2}=\sqrt{23}$

$$(x+4)=\pm\sqrt{23}$$

(5) $x+4=\pm\sqrt{23}$

$$\begin{array}{r} -4 \quad -4 \\ \hline x=-4\pm\sqrt{23} \end{array}$$

(6) $x=-4\pm\sqrt{23}$

Step 1: Add 7 to each side of the equation to put the equation in the form $x^2+bx=c$. Step 2: Since 8 is in the same place as b, b = 8. Add the square of 1/2 of b to each side of the equation. One-half of 8 equals 4, so we are actually adding 4 squared to each side of the equation.
Step 3: We need to fill in the final form in the theorem for completing the square. One-half of b equals 4, so inside the parentheses we have (x + 4). Then we place an exponent of 2 outside the parentheses. Finally, 4 squared equals 16 (4 x 4).
Step 4: First, add 7 and 16 to get 23. Then take the square root of each side of the equation. Taking the square root of a term is the opposite of squaring a term, so we get x + 4 on one side of the equal sign. Remember, every number has a positive and

a negative square root, so we write $\pm\sqrt{23}$. Step 5: Subtract 4 from each side of the equation to isolate the x on one side of the equal sign. The new expression cannot be simplified.
Step 6: The solution to the equation is $x = -4 \pm \sqrt{23}$.

Characteristics that Describe the Graph of a Non-Linear Equation:

A quadratic equation is any equation in the form $y = ax^2 + bx + c$. The graph of a quadratic equation is always a parabola. The vertex of a parabola can be found by putting the quadratic equation in vertex form.

Vertex form of a quadratic equation: $y - k = a(x - h)^2$, where (h, k) is a vertex.

Once a quadratic equation is in vertex form, the vertex is the coordinate point (h, k). If a > 0, then the graph opens up and has a minimum. If a < 0, then the graph opens down and has a maximum.

The roots of a quadratic equation are the solutions of the quadratic equation when y = 0. The roots are the points where the graph intersects the x-axis; therefore, y = 0. The axis of symmetry is the line that passes through the vertex and splits the graph directly in half such that each side is the mirror image of the other. This line is represented by the equation x = h.

Example 7: What are the characteristics of the graph of $y = -x^2 + 6x + 4$?
Step 1: Rewrite the equation in vertex form. The simplest way to do this is by completing the square.

(A)

$$\begin{array}{rcl} y = -x^2 + 6x + 4 \\ \underline{-4 \qquad\qquad -4} \\ y - 4 = -x^2 + 6x \end{array}$$

(B)

$$y - 4 = -x^2 + 6x$$
$$y - 4 = -1(x^2 - 6x)$$

(C)

$$y - 4 - (\tfrac{1}{2} \cdot -6)^2 = -1(x^2 - 6x + (\tfrac{1}{2} \cdot -6)^2)$$
$$y - 4 - (-3)^2 = -1(x^2 - 6x + (-3)^2)$$

(D)

$$y - 4 - 9 = -1(x - 3)^2$$
$$y - 13 = -1(x - 3)^2$$

Step 1A: Subtract 4 from each side of the equation.
Step 1B: Factor -1 out of the right side of the equation to make the term positive.
Step 1C: Add 1/2 of the b term (-6) squared to the right side of the equation and subtract 1/2 of the b term from the left side of the equation (we subtract on the left side of the equation because we factored -1 out of the equation in Step 1B and we need to multiply any number by -1 before we can add or subtract it from the left side of the equation). b = -6, so we are actually adding -3 squared to the right side of the equation and subtracting -3 squared from the left side of the equation.
Step 1D: Complete the theorem for completing the square.

Step 2: Determine whether the graph opens up or down. Now that the equation is in vertex form, we can determine that a = -1. Since a < 0, the graph opens down.
Step 3: Determine the vertex of the parabola. The vertex is the point (h, k). h = 3 and k = 13, so the vertex is (3, 13).
Step 4: Determine the axis of symmetry. The axis of symmetry is the line x = h. The axis of symmetry is x = 3.
Step 5: Determine the roots of the equation. The roots of the equation are the values of x when y = 0.

$$(A)\ 0-13=-1(x-3)^2 \qquad (B)\ \frac{-13}{-1}=\frac{-1(x-3)^2}{-1}$$
$$13=(x-3)^2$$

$$(C)\ \sqrt{13}=\sqrt{(x-3)^2} \qquad (D)\ \pm\sqrt{13}=x-3$$
$$\pm\sqrt{13}=x-3 \qquad\qquad \underline{+3 \qquad +3}$$
$$3\pm\sqrt{13}=x$$

Step 5A: Substitute 0 in place of y.
Step 5B: Divide each side of the equation by -1.
Step 5C: Take the square root of each side of the equation.
Step 5D: The roots of the equation are $3\pm\sqrt{13}\ \left(\text{also written } 3+\sqrt{13} \text{ and } 3-\sqrt{13}\right)$.

If the 13 had been negative, the equation would have had no real roots.

The characteristics of the graph of $y=-x^2+6x+4$ are:

(1) opens down
(2) vertex is (3, 13)
(3) axis of symmetry is: x = 3
(4) roots are
$3\pm\sqrt{13}\ \left(\text{also written } 3+\sqrt{13} \text{ and } 3-\sqrt{13}\right)$.

Spatial Relationships - C

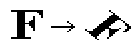
Spatial relationships include understanding geometric transformations as well as recognizing the projection of a three-dimensional object into two dimensions.

Transformations:

If a figure has changed its size, direction, or position, then the figure has been transformed. The following examples will discuss four different transformations.

Rotation of a figure:

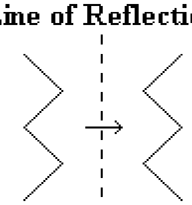
A rotation of a figure changes the direction that a figure is facing. The new figure is found by rotating the original figure about a fixed point for a given number of degrees. The fixed point may be located on or off the original figure.



Reflection of a figure:

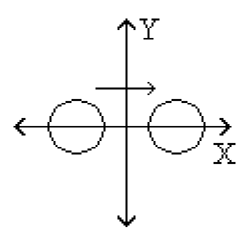
If two figures are reflections, then they are mirror images of each other. A line of reflection can be drawn between the figures such that if the paper upon which the figures had been drawn was folded on this line, the two figures would coincide. That is, all of the points of one figure would lie upon all of the points of the second figure. If any parts of the figures do not coincide, the figures are not reflections of each other.

Line of Reflection



Translation of a figure:

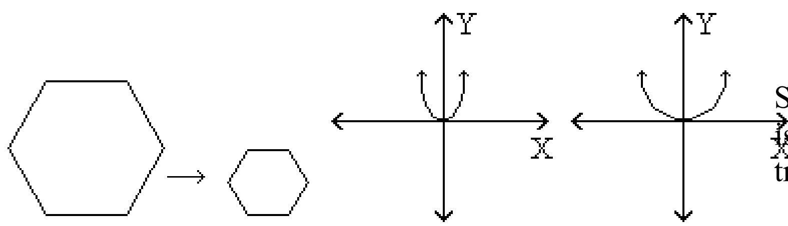
A translation takes a figure and moves it in its entirety along a line from one position to another. Note that after a figure is translated, the figure does not change the direction it is facing, its size, or its shape; only its location has been changed. Watching a train engine move shows a translation in action. As the engine passes you and moves down the track, it changes position, but the size and shape of the train engine do not change. In the following example, the circle has been translated because its position has changed.



Dilation of a figure:

A dilation of a figure can be thought of as the transformation that shrinks or stretches a

figure. In the following examples, the hexagon has shrunk in its size whereas the parabola has become "wider."



How to determine the type of transformation:

To determine the type of transformation that has taken place, ask the following questions:

(1) Has the figure changed its size?
If yes, the transformation is probably a dilation.

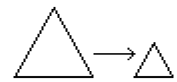
(2) Has the figure changed its position by travelling along a linear (straight) path?
If yes, the transformation is probably a translation.

(3) Do the figures appear to be mirror images of each other?
If yes, the transformation is probably a reflection.

(4) Has the direction the figure was facing changed?
If yes, the transformation is probably a rotation.

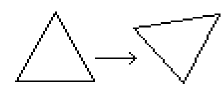
It is possible to combine different types of transformations to make changes to the original figure. The following examples will only involve one transformation.

Example 1: Which transformation was performed on the following figure?



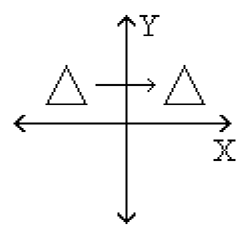
Since the figure has changed its size by getting smaller, the transformation is a <u>dilation</u>.

Example 2: Which transformation was performed on the following figure?



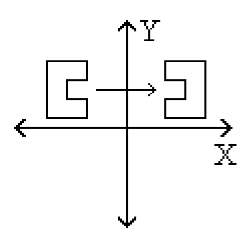
Since the figure has not changed shape, but is facing a different direction, the transformation is a <u>rotation</u>.

Example 3: Which transformation was performed on the following figure?



Since the triangle has shifted to the right along the x-axis, the transformation is a <u>translation</u>.

Example 4: Which transformation was performed on the following figure?



Since the figure appears to have been "flipped" across the y-axis to get its mirror image, the transformation is a <u>reflection</u>.

Spatial Relationships (Projection of a

three-dimensional object into two dimensions:

The following problems will require the interpretation of two-dimensional pictures drawn of a three-dimensional building. Two pictures of either the top, frontal, right, or left view will be presented and then questions will be asked about what a viewer sees from one of the other sides.

A top view that looks like figure A below reflects a building that has one change in the height of the building. Figure B shows a building that has two changes in the height of the building.

Top View

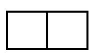


Figure A

Top View

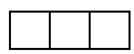
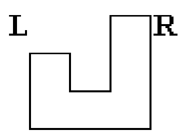


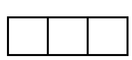
Figure B

A frontal view such as the one in the diagram below better shows the differing heights of the building. In this example, the building is shown to have three different heights from middle to low to high.

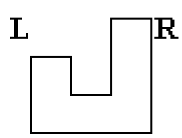


Frontal View

The following are the front and top views of a building to be used for Examples 5 and 6.



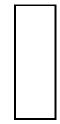
Top View



Frontal View

Example 5: What is the view from the right side of the building?

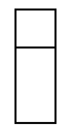
From the right side of the building, the viewer can only see the highest wall. The viewer has no idea if there are any differing heights on the other side of the wall because the view is blocked by the high wall. Thus, the view from the right side of the building is:



Right View

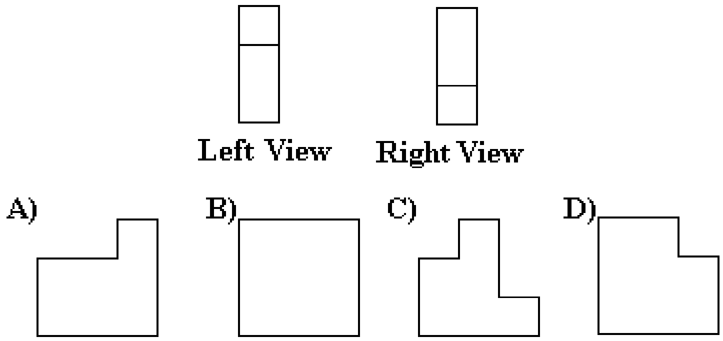
Example 6: What is the view from the left side of the building?

From the left side of the building, the viewer can see two different heights of the building, the middle and highest heights. The "dip" in the building's height cannot be seen by the viewer. Thus the view from the left side of the building is:



Left View

Example 7: The following are the right and left views of a building. Which of the following could be the frontal view?



The right view indicates to the viewer that there is at least one change of height with the known change of height at a lower part of the building. The left view indicates to the viewer that there is at least one change of height with the known change of height at a higher level of the building. The viewer cannot tell whether there is more than one change of height because a top view is not presented to confirm the exact number of height changes. Thus the only choice among the four options is Choice C. Note that Choice A does show a height change on the left side, but shows no height change on the right side. Choice B shows no height changes on either the right or left sides. Choice D shows no height change on the left side.

Functions/Relations - B
A <u>relation</u> can be expressed as a set of ordered pairs such as $\{(3, 1), (2, 3), (-1, -2)\}$. A <u>function</u> is a special type of relation.

There are a few ways to determine if a relation is a function. One way is to look at all of the ordered pairs of a relation. If no two of these ordered pairs have the same x - term (abscissa), then the relation is a function. Another method involves graphing the ordered pairs of a relation on a coordinate graph. If no vertical line crosses this graph at more than one point, then the relation is a function.

A linear function is a function whose graph is a line or subset of a line which is not vertical. A special type of linear function is a constant function, whose graph is a horizontal line or subset of a horizontal line.

Using these guidelines, find the value for q that will make the following relation not a function: $\{(16, 3), (q^2, 7)\}$.

The answer is 4 because if g = 4, then the second ordered pair would be (16, 7). The two ordered pairs would have the same x - term, and for that reason the relation would not be a function.

Functions: Notation
A <u>relation</u> is a set of ordered pairs such as $\{(3, 1), (2, 3), (-1, -2)\}$. A <u>function</u> is a relation in which no two ordered pairs of a set have the same x-value. Functions are often abbreviated as f(x) = y. This is read, "the value of f at x is y."

Given y = 3x - 2, you can find the value of y when x = -3 by substituting -3 for x.

The value of y when x is -3 is -11.

The <u>domain</u> of a function is the set of all x-values of the ordered pairs in the function. The <u>range</u> of a function is the set of all y-values. To find the range of a function when given its domain, follow the substitution method above.

Example 1: Find the range of f(x) = 3x - 2 given that the domain is $\{0, 1, 2\}$.
<u>Solution</u>: Substitute each number in the domain into the function f(x) = 3x - 2 and simplify to determine the numbers in the range.

```
f(0) = 3(0) - 2         f(1) = 3(1)
- 2         f(2) = 3(2) - 2
f(0) = 0 - 2      f(1) = 3 - 2
```

f(2) = 6 - 2
f(0) = -2 f(1) = 1
f(2) = 4

From the domain, it can be determined that the range is {−2,1,4}.

Example 2: Find f(3) + h(4), when given f(x) = 3x - 1 and h(x) = 2x + 3.

1. Substitute 3 in f(x) = 3x - 1 to get 8
2. Substitute 4 in h(x) = 2x + 3 to get 11
3. Now add f(3) + h(4) = 8 + 11 = 19

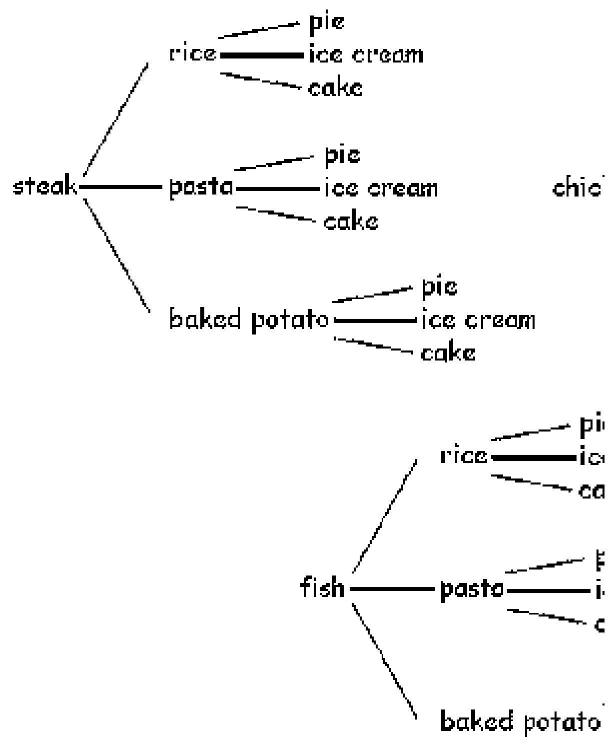
Probability

Probability is the measure of the chance that a specific outcome will occur. Probability methods at this level include using tree diagrams, sample space, the fundamental counting principle, adding and multiplying probabilities for independent and dependent events, calculating expected value, conditional probability, experimental probability, and theoretical probability.

Tree diagrams are probability tools which represent possible outcomes. If you went to dinner at a banquet, you may be presented with the following possibilities:

Main dishes: steak, fish, chicken
Side dishes: rice, pasta, baked potato
Dessert: pie, ice cream, cake

Suppose that at this dinner you were asked to choose one item from each category: one main dish, one side dish, and one dessert. How many different possible meals could you choose? A tree diagram which gives the sample space (the choices) would help you quickly count the choices:



The above illustration is a tree diagram. All that is left to do is to count the choices down the right side of each branch: there are 27 different possible meals.

Two events are independent if the probability of one event happening has no influence on the probability of the other event happening. If you roll one die and toss one coin, you know that the number on the die has nothing to do with whether the coin toss results in heads or tails. The formula for determining the probability that two independent events will occur is below.

P(A and B) = Probability of A x Probability of B = P(A) x P(B)

Example 1: What is the probability a coin toss resulting in heads and a roll of the die resulting in a 3 or less?

(1) P(A and B) = P(A) x P(B)
(2) P(A and B) = 1/2 x 3/6
(3) P(A and B) = 1/2 x 1/2
(4) P(A and B) = 1/4

Step 1: Choose the correct formula for the probability of A and B happening.

Step 2: There are only two possibilities when tossing a coin (heads and tails), so the probability of a coin toss resulting in heads is 1/2. There are six possibilities when rolling a die (1, 2, 3, 4, 5, and 6). Only three of those possibilities are equal to or less than 3, so the probability of the roll of a die resulting in a 3 or less is 3/6. Substitute the probabilities into the formula.
Step 3: Reduce the fractions before multiplying.
Step 4: 1/2 times 1/2 equals 1/4. Remember to multiply numerators and denominators straight across.

Answer: 1/4

Dependent events are events which influence one another's probability of occurring. The formula for determining the probability that two dependent events will occur is below.

P(A and B) = Probability of A x Probability of B given A = P(A) x P(B, given A)

Example 2: If you draw one card from a deck, put it aside, and then draw another card, what is the probability that each card drawn is a heart?

(1) P(A and B) = P(A) x P(B, given A)
(2) P(A and B) = 13/52 x 12/51
(3) P(A and B) = 156/2652
(4) P(A and B) = 13/221

Step 1: Choose the correct formula for the probability of A and B happening.
Step 2: There are 52 cards in a deck of cards. 13 of the cards in each deck are hearts. The probability that the first card drawn is a heart is 13/52. The probability that the second card drawn is a heart is 12/51 because there is one less heart in the deck and one less card in the deck. Substitute the probabilities into the formula.
Step 3: Multiply the fractions. Remember to multiply numerators straight across and denominators straight across.
Step 4: Reduce the fraction completely.

Answer: 13/221

The formula for calculating expected value is:
(E = result of outcome #1 x probability of a outcome #1 + result of outcome #2 x probability of outcome #2).
Businesses can use such a formula to roughly project expected profits under specific conditions.

Example 3: Suppose you owned a snack bar at a beach. Let's say that in a good summer you make $3,000 and in a bad summer you lose $50. The greatest determining factor of a good or bad year has been the weather, and all indications show that the approaching summer season has an 89% chance of being sunny and warm - a good year. What is your projected profit for the approaching season?

(1) E = ($3,000 x 0.89) + (-$50 x 0.11)
(2) E = ($2,670) + (-$5.50)
(3) E = $2,664.50

Step 1: The result of a good summer is $3,000 and the probability that there will be a good summer is 89% (0.89). The result of a bad summer is losing $50 (-$50) and the probability that there will be a bad summer is 11% (0.11). Use these values to fill in the formula for calculating the expected value.
Step 2: Multiply $3,000 by 0.89 to get $2,670 and multiply -$50 by 0.11 to get -$5.50.
Step 3: Add the results of Step 2.

The expected profit for the approaching summer season is $2664.50.

To calculate conditional probability, you must find the probability of an event based on the fact that another event has already happened.

Example 4: An algebra class gets a new student, a girl. This new student happens to have two younger siblings. Find the probability that one of the new student's siblings is also a girl.

Solution: Examine all of the possible ways three siblings might be arranged in terms of their gender. The fact that the first sibling, the new girl in class, is a girl alters the possible choices for the problem. The possibilities for the genders of 3 siblings are: GGG (Girl, Girl, Girl), GGB, GBG, GBB, BGG, BGB, BBG, BBB. From these possibilities, you can cancel out any that don't begin with G since we know that the oldest sibling is a girl. That leaves us with 4 possibilities: GGG, GGB, GBG, and GBB. Three of these result in two siblings that are girls. Therefore the probability that at least two of the siblings are girls is 3/4 or 75%.

Experimental probability is a way to predict future events using data from past events. Experimental probability is calculated by dividing the number of occurrences of an event by the number of trials of an experiment. A football coach, for example, can predict how well his receiver will complete passes. If the receiver has been completing 10 out of every 25 passes thrown to him, then the coach can use experimental probability to predict how well he will complete passes in the next game: 10/25 = 0.4 or 40%. The prediction is that the receiver will complete 4 out of 10 or 2 out of 5 passes thrown to him.

In contrast, theoretical probability or mathematical probability refers to finding the probability of an event before any trials of an experiment have been performed. Often theoretical or mathematical probability is referred to as just probability.

If you want to find the probability of rolling a die and getting a 4, you simply set up the fraction 1/6 (1 because there is only one 4 on the die and 6 because there are six sides on the die meaning six different possible outcomes.) Therefore, before we even roll a die, we know that theoretical probability tells us that we have a 1 in 6 chance of rolling a 4.

The Counting Principle

Counting the choices involves determining how many choices are available in a given situation. If there are A choices for one way and B choices for another, then the total number of choices is A x B.

Example 5: Ana went to the world's largest amusement park. There were 10 different rides, 14 roller coasters, 8 shows, and 6 shops. How many different ways can Ana see all of the attractions?

Solution: Multiply 10 x 14 x 8 x 6 = 6,720

Answer: 6,720 ways

Statistics

Statistics is the study of numerical data. It can be collected, assembleed, and classified so as to present meaningful information. Statistical data can be presented in many different forms: stem and leaf displays, frequency tables, quartiles, variance, etc.

Stem and leaf displays show large numbers in a concise format. This display is based on place value. The following stem and leaf display compares test scores for two algebra classes.

The scores in Ms. Williams class are:

65, 67, 71, 73, 75, 75, 76, 76, 77, 78, 79, 85, 89

The scores in Mr. Ruiz's class are:

60, 63, 67, 75, 75, 76, 76, 81, 81, 84, 85, 86, 88

Ms. Williams		Mr. Ruiz
5, 7	- 6 -	7, 3, 0
1, 3, 5, 5, 6, 6, 7, 8, 9	- 7 -	6, 6, 5, 5
5, 9	- 8 -	8, 6, 5, 4, 1, 1
1	- 9 -	4

The first digit of the score is located in the center column. The second digits are listed beside each of the first digits. For instance, the lowest scores in Ms. Williams class are 67 and 65.

The stem and leaf display also allows you to see the scores comparatively. We can clearly see that Ms. Williams had many scores in the 70's and that Mr. Ruiz had many scores in the 80's.

Frequency tables present outcomes of experiments in a format which shows the frequency (the amount of times) each outcome occurs. The following terms are important in understanding frequency tables.

Frequency: number of occurrences
Relative Frequency: derived by dividing the frequency by the total number of measurements
Cumulative Frequency: the number of measurements that are less than or equal to a given value
Total Frequency: the total number of occurrences
Interval: the range of measurements

The frequency table below gives data for the number of goals Melissa kicked per game during one season.

Interval	**Frequency**	**Relative Frequency**	**Cumulat Frequen**
1	**6**	$\frac{6}{20}$	**6**
2	**5**	$\frac{5}{20}$	**11**
3	**3**	$\frac{3}{20}$	**14**
4	**2**	$\frac{2}{20}$	**16**
5	**2**	$\frac{2}{20}$	**18**
6	**1**	$\frac{1}{20}$	**19**
7	**1**	$\frac{1}{20}$	**20**

Total Frequency: 20

Intervals: 7

From this table, we can see that there were between 5 times that Melissa scored 2 goals. In other words, the frequency with which she scored 2 goals was 5 times during the season this table covers.

Quartiles are associated with median measures. A median is the middle number or, in an even list of numbers, the average of the two middle numbers. The list needs to always be in ascending or descending order. The median of 0, 1, 3, 5, 7, 9 is 4 because 3 + 5 = 8 and 8 ÷ 2 = 4. Three quartiles (Q1, Q2, and Q3) divide such data into four equal groups, provided the data is listed from smallest to largest.

Q1 is the median of the smallest 50% of the numbers.
Q2 is the median of the entire list of data.
Q3 is the median of the largest 50% of the numbers.

Find Q1, Q2, and Q3 for the following list of 20 numbers:

21, 23, 24, 26, 31, 37, 39, 45, 46, 49
52, 61, 68, 75, 76, 78, 81, 85, 91, 98

$$\text{Step 1}: 49+52=101 \qquad \frac{101}{2}=50.5 \qquad Q2=50.5$$

$$\text{Step 2}: 31+37=68 \qquad \frac{68}{2}=34 \qquad Q1=34$$

$$\text{Step 3}: 76+78=154 \qquad \frac{154}{2}=77 \qquad Q3=77$$

Step 1: Find Q2 by averaging 49 and 52, the two middle numbers since there is an even amount.
Step 2: Find Q1 by calculating the median of the smallest 50% of numbers.
Step 3: Find Q3 by calculating the median of the largest 50% of numbers.

Variance describes the average of the squares of the deviations of all measurements from their mean. In other words, the variance gives information about how close each element of a series of numbers is to the mean of the series of numbers.

You are probably familiar with finding the mean (average) of numbers. At times it is important to find not only the mean, but how a series of numbers compares to their mean. For instance, list A = 4, 6, 7, 8, 11 and list B = 2, 3, 8, 11, 12. Both have means of 7.2, but the mean doesn't tell us anything about the variance (how close each of the numbers is to the mean).

The following example shows how to find the variance for List A.

(1)	**(2)**
4 - 7.2 = -3.2	**-3.2 x -3.2 = 10.24**
6 - 7.2 = -1.2	**-1.2 x -1.2 = 1.44**
7 - 7.2 = -0.2	**-0.2 x -0.2 = 0.04**
8 - 7.2 = 0.8	**0.8 x 0.8 = 0.64**
11 - 7.2 = 3.8	**3.8 x 3.8 = 14.44**

+

(4)	**(5)**
26.80 ÷ 5 = 5.36	**List A: 5.36**
	List B: 16.56

Step 1: Find the deviations from the mean

for each number in List A. Subtract the mean (7.2) from each number in the list.
Step 2: Square each of the deviations.
Step 3: Add the squared numbers together.
Step 4: Divide the total by the number of addends (numbers added together). The variance for List A is 26.80 ÷ 5 = 5.36.
Step 5: Compare the results for List A and List B.

Finding the variance has shown us that the numbers in List B are scattered further from the mean than the numbers in list A.

Mode is a statistical measure that describes which number(s) occur(s) the most in a set of data. There can be 0, 1, or more than one mode.

Example: Find the mode(s) of the following set of numbers.

97, 100, 65, 80, 97, 72, 63, 80

Answer: 97 and 80